

# DYNAMIC MODELLING OF TREE FORM

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**ABSTRACT.** Tree profile or taper models are an important component of decision support systems for forest management. An extensive body of theory and observations on the mechanisms of tree form development has accumulated over the last 150 years. Quantitative mensurational models, however, have made little use of that information. Taper models remain largely empirical and static, describing tree dimensions and shape at one point in time. Here taper equations are derived from simplified models of radial stem growth consistent with physiological knowledge. It is expected that this approach may improve accuracy, especially when stands are subjected to varied density management alternatives.

**Keywords:** taper, tree growth, pipe model

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## 1 INTRODUCTION

Tree taper or profile models are an important component of decision support systems for forest management (Husch et al. 2003, Laar and Aka 2007, Burkhart and Tomé 2012). Consisting of equations relating stem diameter or cross-sectional area to height above ground, they have a number of uses, notably the disaggregation of stand and tree measurements or predictions into yields by product types and dimensions. Emphasis lies on accuracy, flexibility, and simplicity. Generally the models are empirical, based on large data sets, and static, describing tree form at a single point in time.

At a more fundamental level, mechanisms of tree form development have been studied for a long time (Gray 1956, Larson 1963). Two general theories, with variations and elaborations, are the most popular. Metzger in 1893 sustained that tree shape optimizes resistance to bending by the wind. In its simplest form the theory predicts a cubic paraboloid for the form of the branch-free part of the bole. For elaborations on these principles see, for instance, Gaffrey and Sloboda (2001). The other main view, introduced by Pressler in 1864, assumes that the increment in cross-section at a certain height up the stem is proportional to the amount of foliage above that height. The basic idea can be traced back to Leonardo da Vinci (Mendes France 1981, Aratsu 1998, Sone et al. 2009, Eloy 2011). The main bole would approach a quadratic paraboloid. Pressler's ideas were re-developed by Japanese scientists in the 1960's, and under the name of the *pipe model theory* have received much attention from plant physiologists. Examples of models derived

from such principles can be seen in Chiba (1990), Renolls (1994), Houllier et al. (1995), Deleuze and Houllier (2002), Mäkelä (2002), Valentine et al. (2012). The co-existence of the theories of Metzger and of Pressler, despite of predicting different diameter-length relationships, points to a large natural variability that makes difficult to reach definite conclusions about fundamental mechanisms. The theories address mainly shape on the branch-free bole above the basal swelling; stem form within the crown is less-often defined precisely, and the butt swell is largely ignored.

We link both research streams, developing taper models suitable for practical management use that are consistent with mechanistic models of stem development. The aim is to obtain tractable dynamic models for the whole stem, with a realistic representation of the evolution of both within-crown and basal bole shapes. Simplifications of a general model are introduced to obtain explicit mathematical solutions.

## 2 MODELS

**2.1 Stem form development.** Trees produce a layer of wood each year, and these layers accumulate on top of each other to determine the stem form at any given time. First, we model how the annual layer thickness varies with distance from the tree top. According to Pressler, thickness measured as growth-ring cross-sectional area  $\Delta s$  (or volume increment per unit length) increases monotonically with distance  $x$  down to the base of the green crown, below which it remains constant. A commonly-used approximation as-

sumes that  $\Delta s$  increases linearly with  $x$  within the crown, so that the annual area increment has the form of a piece-wise linear ramp with a horizontal maximum (e. g., Mitchell 1975). Our model admits more general “rounded ramps”, and includes also a basal swelling that depends on distance from the ground.

A second assumption is that, ignoring butt swell, the relative growth pattern remains the same over time, just moving upwards as the tree grows in height. That is, the  $\Delta s$  vs.  $x$  functions for different layers are proportional. Under Pressler’s model this is not exactly true if crown length changes, but the assumption is necessary for obtaining closed-form solutions, and as discussed later, it may be expected to produce acceptable results.

Mitchell (1975) describes well this model and its consequences, for a linear ramp and no butt swell, and his Figure 4 is particularly instructive. The increment function translational invariance is behind the stability of the classic Type 3 growth ring sequences of Duff and Nolan (1953). They measured growth rings at the center of each internode, mostly within crowns. The equations below can be seen as continuous analogs of the Duff and Nolan (1953) sequences, extended to full tree lengths and simplified by using areas instead of radial increments.

Fig. 1 shows periodic area increments for Douglas-fir stem-analysis data from British Columbia. These correspond to the thickness of 5 consecutive growth layers, a number chosen because of the low growth rates in these stands and to reduce measurement error noise. The observations suggest a more gradual growth rate change than the one predicted by the piece-wise linear Pressler ramp function. The higher growth rates near the ground that cause butt swelling are clearly seen. Similar patterns have been observed for other species.

Typical section area profiles are shown in Fig. 2. These are generated by accumulation of the growth rates in Fig. 1. Variability is much higher in Fig. 1 than in Fig. 2, reflecting the fact that noise tends to be amplified by differentiation and reduced by integration. Therefore, from a practical point of view, accuracy in modelling growth rates may not be overly critical for predicting tree profiles or volumes.

**2.2 Mathematical formulation.** Let  $s$  be the stem cross-sectional area at a height  $h$  above ground when the total tree height is  $H$ . This is a function  $s = s(h, H)$ . In Pressler’s model the increment of  $s$  is proportional to the length of green crown above  $h$ . Then, expressing increments relative to height growth, the growth rate at a fixed  $h$  is

$$\frac{\partial s}{\partial H} \propto \varphi(x) = \varphi(H - h) ,$$

where  $x = H - h$  is distance from the top, and  $\varphi(x) = \min\{x/c, 1\}$  for a green crown length  $c$ . This  $\varphi$  is a

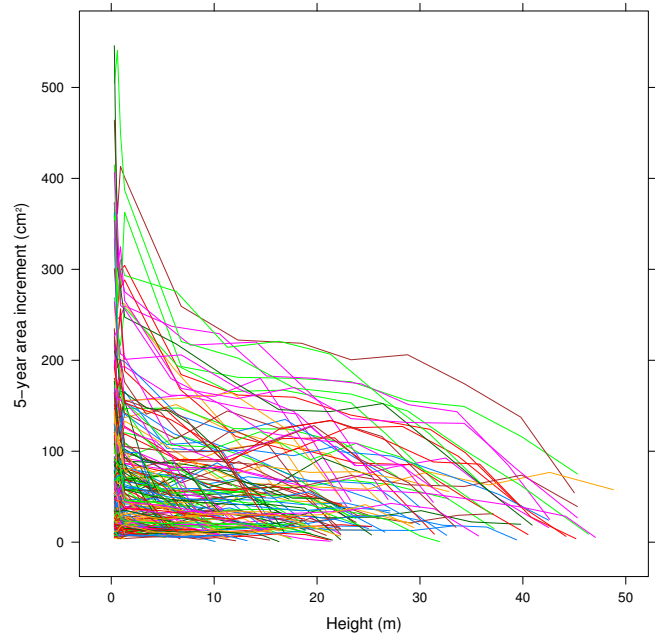


Figure 1: 5-year cross-sectional area increments for Douglas-fir trees as functions of height up the stem. Each curve corresponds to 5 contiguous growth layers selected at random from each of 222 trees. Area increases with distance from the top until becoming roughly constant below the crown base, increasing again near the ground.

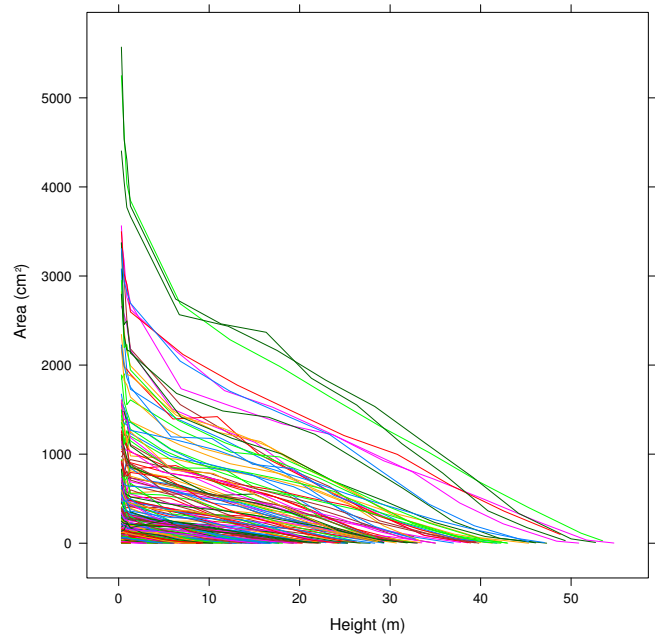


Figure 2: Douglas-fir cross-section area profiles. One profile selected at random from each of 222 trees.

piece-wise linear ramp, directly proportional to  $x$  for  $x \leq c$  and equal to 1 for  $x \geq c$ . As indicated before, we shall consider more general forms of  $\varphi$ , but to simplify will neglect any changes of  $c$  and of the proportionality factor over time. Specifically, it is assumed that  $\varphi(x)$  is a non-decreasing function of  $x$  with  $\varphi(0) = 0$  and  $\varphi(\infty) = 1$ . In addition, we include an additional buttswell increment component that depends on the distance from the ground:

$$\frac{\partial s}{\partial H} = b_0[\varphi(H - h) + \eta(h)]. \quad (1)$$

The asymptote parameter  $b_0$  may vary from tree to tree.

Observed tree growth rate functions  $\partial s/\partial H$  are illustrated in Fig. 1. Near the tree top the increment is dominated by  $\varphi$ , while near the base the butt swell represented by  $\eta$  becomes important.

To obtain the profile function  $s(h, H)$  we integrate over  $H$  for a fixed  $h$ . Growth starts when  $H$  reaches the level  $h$ , so integrating from  $H = h$  to  $H = H$ ,

$$\begin{aligned} s(h, H) &= b_0 \int_h^H [\varphi(H - h) + \eta(h)] dH \\ &= b_0 \left[ \int_0^{H-h} \varphi(x) dx + (H - h)\eta(h) \right]. \end{aligned}$$

Or, writing  $\int_0^y \varphi(x) dx \equiv \Phi(y)$ ,

$$s(h, H) = b_0[\Phi(H - h) + (H - h)\eta(h)]. \quad (2)$$

Knowing the under-bark dbh, and therefore the cross-section  $S$  at  $h = 1.3\text{m}$ , the tree-specific  $b_0$  can be eliminated:

$$s(h, H, S) = S \frac{\Phi(H - h) + (H - h)\eta(h)}{\Phi(H - 1.3) + (H - 1.3)\eta(1.3)}. \quad (3)$$

Typical profiles corresponding to eq. (2) are shown in Figure 2.  $\Phi(x)$  increases rapidly near the tree top, later approximating a straight line as  $\varphi$  reaches or approaches an asymptote. Closer to the base there is again a curvature due to the butt swell.

### 2.3 Modelling $\varphi$ . Pressler’s model is

$$\begin{aligned} \varphi(x) &= \min\{x/b_1, 1\} = 1 - \max\{1 - x/b_1, 0\} \\ &\equiv 1 - (1 - x/b_1)_+, \end{aligned}$$

where  $b_1$  is a parameter related to crown length. Fig. 1, however, suggest something closer to an exponential

$$\varphi(x) = 1 - \exp(-x/b_1),$$

or even a shallower hyperbolic

$$\varphi(x) = \frac{x}{x + b_1} = 1 - \frac{1}{1 + x/b_1}.$$

This might be sufficient, but for greater flexibility we shall use a more general model characterized by a variable shape parameter  $b_2$ . Ideally, we want a model of the right shape that could be integrated analytically to obtain a closed form expression for  $\Phi$ . A suitable choice is

$$\varphi(x) = 1 - \delta_{b_2}(x/b_1), \quad (4)$$

where

$$\delta_\theta(u) = (1 - \theta u)_+^{1/\theta} \quad (5)$$

with  $\theta \neq 0$ . Here  $\delta_\theta$  is called a “decay function”, and  $1 - \delta_\theta$  is a ramp. For the special case  $\theta = 0$  we define  $\delta_0$  as the limit

$$\delta_0(u) = \lim_{\theta \rightarrow 0} (1 - \theta u)_+^{1/\theta} = \exp(-u), \quad (6)$$

which gives the exponential model above<sup>1</sup>. Other special cases include Pressler’s, with  $\theta = 1$ , and the hyperbolic  $\theta = -1$ . Fig. 3 shows some examples.

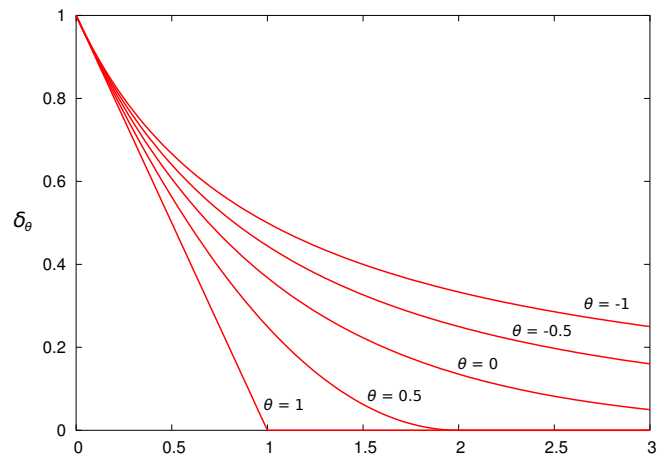


Figure 3: Examples of decay functions  $\delta_\theta$ , eq. (5)–(6).

Integrating,  $\Phi$  in eq. (2)–(3) is found to be

$$\Phi(x) = x - \frac{b_1}{b_2+1} \left[ 1 - \delta_{b_2/(b_2+1)}\left(\frac{b_2+1}{b_1}x\right) \right] \quad (7)$$

for  $b_2 \neq -1$ . If  $b_2 = -1$  then  $\Phi(x) = x - b_1 \ln(x/b_1 + 1)$ .

**2.4 Butt swell.** The additional butt-swell increment in eq. (1) decreases with distance from the base, and can be modelled by a suitably scaled decay function:

$$\eta(h) = b_3 \delta_{b_5}(h/b_4). \quad (8)$$

<sup>1</sup>  $\delta_\theta$  is the inverse of a Box-Cox transformation, used to define a family of growth or probability distribution functions by García (2008). Eq. (4) corresponds to the functions with no inflection point obtained by fixing one of the shape parameters at 1.

**2.5 Exponential model.** The final profile model in terms of cross-section area is defined by equations (3), (7), (5) and (8). The case  $b_2 = b_5 = 0$  may be of interest as a simpler and parsimonious 3-parameter model:

$$s(h, H, S) = S \frac{s(h, H)}{s(1.3, H)} \quad (9)$$

$$s(h, H) \propto H - h - b_1 + b_1 \exp[-(H - h)/b_1] + b_3(H - h) \exp(-h/b_4). \quad (10)$$

The *modified Brink function* of Gadow and Hui (1999) and Hussein et al. (2008) is a similar but not identical combination of exponentials, with a different motivation, that has been successful in a number of applications.

Both this and the general model can be integrated to obtain an explicit expression for the volume between any two height levels  $h_1$  and  $h_2$ . However, inverting to obtain the height for a given diameter must be done numerically.

### 3 EXAMPLE

Models were fitted to full stem-analysis Douglas-fir data collected in the University of British Columbia research forests, kindly provided by Prof. Valerie LeMay. The ring measurement data were structured into growth layers, describing a tree profile at a point in time. Layer tips were interpolated with Carmean's method (Dyer and Bailey 1987).

For each tree with breast-height measurements, one layer was chosen at random among those with complete data and taller than breast height. This gave 222 profiles, with a total of 2028 measurements (Fig. 2). Equation (3) was fitted to the observed cross-section areas by nonlinear least-squares. R code is shown the Appendix. Model versions with and without free shape parameters  $b_2$  and  $b_5$  were tried. Table 1 shows the parameter estimates.

The tree-top shape parameter  $b_2$  was not significantly different from 0, and contributed little to a better fit. The opposite was true for the butt-swell shape parameter  $b_5$ . Therefore, the version with an exponential decay function for the top and a general or hyperbolic ( $b_5 \approx -1$ ) decay function for the butt swell seems appropriate in this instance.

The exponential-hyperbolic model is

$$s(h, H, S) = S \frac{s(h, H)}{s(1.3, H)} \quad (11)$$

$$s(h, H) \propto H - h - 2.569\{1 - \exp[-(H - h)/2.569]\} + \frac{1.042(H - h)}{1 + h/0.3012}. \quad (12)$$

Figures 4 and 5 show the annual growth layers generated by this model for one sample tree. Annual increments

are consistent with the expected tree form development over time. The linear area-length relationship observed over much of the stem in Fig. 4, and also in Fig. 2, corresponds to the *taper line* extensively documented by Gray (1956) and others.

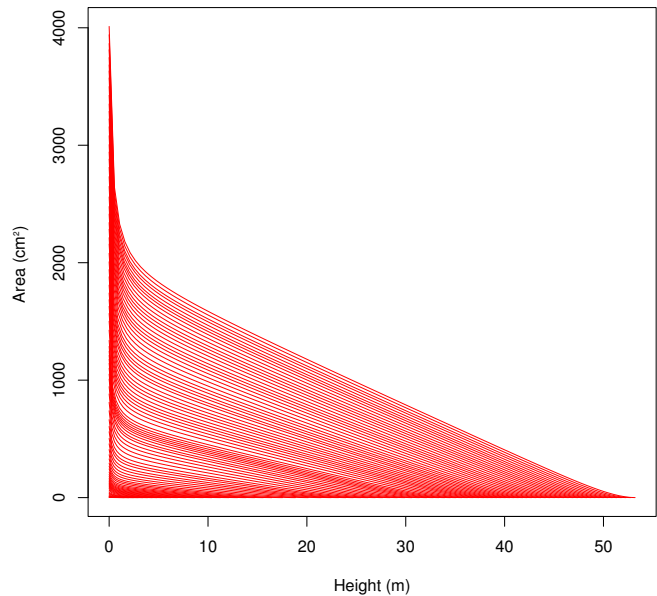


Figure 4: Model cross-sectional areas of annual growth ring layers for one tree.

A simple nonlinear least-squares procedure was used here. A detailed discussion of estimation methods for taper equations is outside the scope of this article, but a few comments may be appropriate. E. g., instead of cross-sectional areas as dependent variables one could use diameters, which would give a different weighting to the data points. Similarly, it is often advocated to give lower weighting to the larger values, to account for the heterocedasticity generated by accumulated errors. On the other hand, the larger diameters or areas are usually the most important in the applications, and an analysis using loss functions would suggest an opposite weighting.

Another estimation issue is the autocorrelation of consecutive measurements in a same tree, and as in the analysis of growth curves, more complex error structures have been introduced to deal with this (e.g., Gregoire and Schabenberger 1996, Williams and Reich 1997). Note however that with  $m$  measurements in each of  $n$  individuals, the number of potentially correlated pairs of measurements is  $nm(m - 1)/2$  out of a total of  $nm(nm - 1)/2$  pairs. That is, a proportion  $(m - 1)/(mn - 1) \approx 1/n$ , usually a fraction of 1%, so that it seems rather unlikely that modelling those correlations could make much difference. The same is true in many longitudinal data studies.

Table 1: Taper function parameter estimates.

Model	$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$RSE$
Exponential	2.977	0 (fixed)	0.6114	0.9206	0 (fixed)	73.64
$b_2 \neq 0$	4.599	0.6305 ( $p = 0.3$ )	0.6155	0.9263	0 (fixed)	73.58
$b_5 \neq 0$	2.522	0 (fixed)	1.170	0.2398	-1.133 ( $p = 10^{-7}$ )	72.83
Full	5.068	1.062 ( $p = 0.4$ )	1.202	0.2282	-1.162 ( $p = 7 \cdot 10^{-8}$ )	72.74
Exp-Hyp	2.569	0 (fixed)	1.042	0.3012	-1 (fixed)	72.82

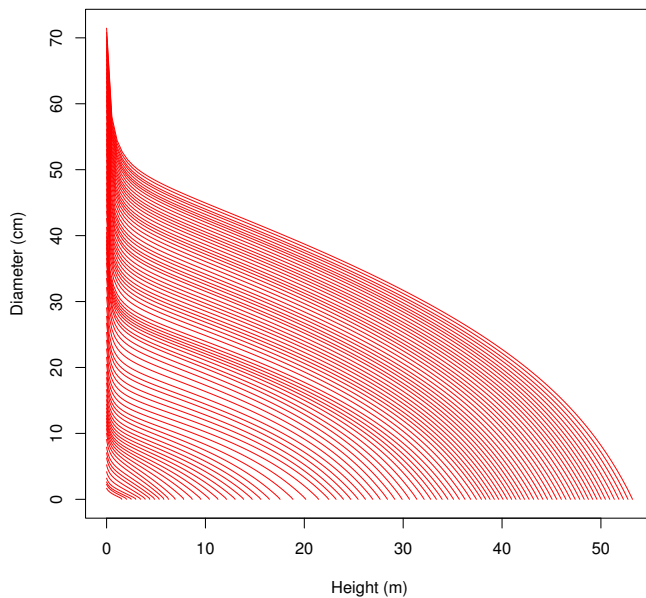


Figure 5: Model diameters of annual growth ring layers for one tree.

Potentially more important are form differences between stands. The form of a suppressed tree can be expected to differ from that of a dominant tree of the same size in a younger stand. This could be handled by introducing stand-level covariates (Muhairwe et al. 1994), or by hierarchical modelling, at the cost of losing some of the simplicity appeal of taper equations. The present model could aid in such investigations through easily interpretable parameters:  $b_1$  is related to crown length and dominance, while the more pronounced butt swelling that has been associated to dominant trees is described by  $b_3$  and/or  $b_4$ .

#### 4 CONCLUSIONS

A class of tractable dynamic taper models was derived on the basis of widely accepted biological principles. They produce stem profiles that evolve in a consistent way over time, a property that can be important for some applications.

Conceptually, the assumption of a stable crown length

is not entirely satisfactory, although it seems unlikely that the effects on integrated profile shapes might be practically important. It would be interesting to see if the assumption could be relaxed while still having a closed-form solution.

As conventional taper equations, the models parsimony and realistic representation of stem shape can be an advantage. Preliminary comparisons to other models, not reported here, have been encouraging, but further testing is needed.

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## BIOGRAPHICAL NOTE

Oscar García received a professional forester degree (Ingeniero Forestal) and an M.Sc. in Mathematical Statistics & Operations Research from the University of Chile, and a Ph.D. in Forest Resources from the University of Georgia, USA. Between 1976 and 1992 he worked in New Zealand for the Forest Research Institute, developing forest management planning decision-support systems and being responsible for much of their plantation growth modeling program. Later, he led a Government / Industry cooperative eucalypt modeling project in Chile, and conducted forest planning research in Spain for the Galician Government. Oscar occupied the Chair of Forest Mensuration at the Universidad Austral de Chile, and held Guest Professor positions at the ENGREF in France, KVL in Denmark, and the Universidad Politécnica de Madrid in Spain. He was a Professor and Endowed Chair in Forest Growth and Yield at the University of Northern British Columbia, Canada, between 2000 and 2014.

## APPENDIX: R COMPUTER CODE

Example of model fit:

```
summary(fitexp <- nls(sarea ~ profile(sheight,
  treeHt, treeSarea, b=c(b1, 0, b3, b4, 0)),
  data=profs, start=list(b1=5, b3=1, b4=2)))
```

The following functions are used:

```

profile <- function(h, H, S, b) {
# Taper model giving area at level h
  S * proffree(h, H, b) / proffree(1.3, H, b)
}

proffree <- function(h, H, b) {
# Free profile, gives area at level h for b0=1
  Phi(H - h, b) + (H - h) * eta(h, b)
}

Phi <- function(x, b) {
# Phi function, integral of phi (top taper)
  x - b[1] * (1 - delta((b[2] + 1) * x /
    b[1], b[2] / (b[2] + 1))) / (b[2] + 1)
}

eta <- function(x, b) {
# eta function (butt swell)
  b[3] * delta(x / b[4], b[5])
}

delta <- function(x, theta=0) {
# Decay function
  if(theta == 0)
    exp(-x)
  else
    pmax(1 - theta * x, 0)^(1 / theta)
}

```