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A STRENGTHENING PROCEDURE FOR THE PATH FORMULATION OF THE AREA-BASED ADJACENCY PROBLEM IN HARVEST SCHEDULING MODELS

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Abstract. Spatially-explicit harvest scheduling models optimize the spatiotemporal layout of forest management actions to best meet management objectives such as profit maximization, even flow of products, or wildlife habitat preservation while satisfying a variety of constraints. This investigation focuses on modeling maximum harvest opening size restrictions whose role is to limit the size of contiguous clear cuts on a forested landscape. These restrictions, a.k.a. green-up constraints, allow adjacent forest stands to be cut within a pre-specified timeframe, called green-up period, only if their combined area does not exceed a limit. We present a strengthening procedure for one of the existing integer programming formulations of this so-called Area Restriction Model and test the computational performance of the new model on sixty hypothetical and seven real forest planning applications. The results suggest that the strengthened model can often outperform the other three existing formulations. We also find that the original Path Model is still competitive in terms of solution times.

Keywords: Spatially-explicit harvest scheduling, area-based adjacency, integer programming

1 Introduction

Spatially-explicit harvest scheduling models optimize the spatial layout of forest management actions over time to best meet management objectives such as profit maximization, even flow of products, or wildlife habitat preservation while satisfying a variety of constraints, including maximum harvest opening size restrictions. These models assign various silvicultural prescriptions, such as clear cuts, thinning or shelterwood treatments, to forest management units (see polygons on Fig. 1). Other spatial decisions, road-building being one example, may also be part of harvest scheduling models. Management decisions, such as whether to cut a management unit or not, or whether to build a road link in a particular planning period are typically modeled using 0-1 variables. Harvest scheduling models such as these are thus 0-1 programs. A variety of restrictions, some spatially-explicit and some not, may also be modeled, including timber-flow constraints (e.g., Thompson et al. 1994), target ending age or inventory constraints (e.g., McDill and Braze 2000), or maximum harvest opening size restrictions (e.g., Meneghin et al. 1988), which are the focus of this paper.

Adjacency constraints (a.k.a. green-up or maximum harvest opening size constraints) limit the size of contiguous clear-cuts. These restrictions, which are often part of legal requirements or certification standards in North America (e.g., Barrett et al. 1998, Sustainable Forest Initiative 2010, Boston and Bettinger 2002), have been promoted as a tool to mitigate the negative impacts of timber harvests (e.g., Thompson et al. 1973, Jones et al. 1991, Murray and Church. 1996a, 1996b, Snyder and ReVelle 1996a, 1996b, 1997a, 1997b, Carter et al. 1997, Murray 1999). Although maximum harvest opening size constraints spatially disperse the harvest activities, and thus relieve the landscape from the concentration of this type of human disturbance, they have also been shown to fragment and disperse mature forest habitat (Harris 1984, Franklin and Forman 1987, Barrett et al. 1998, Borges and Hoganson 2000). To mitigate these negative consequences, Rebain and McDill (2003a, 2003b) proposed a 0-1 programming formulation that allows the forest planner to promote or to require the preservation, maintenance or creation of a certain amount of mature forest habitat in large patches over time in models with maximum harvest opening size constraints. A drawback of combining both harvest opening size and mature

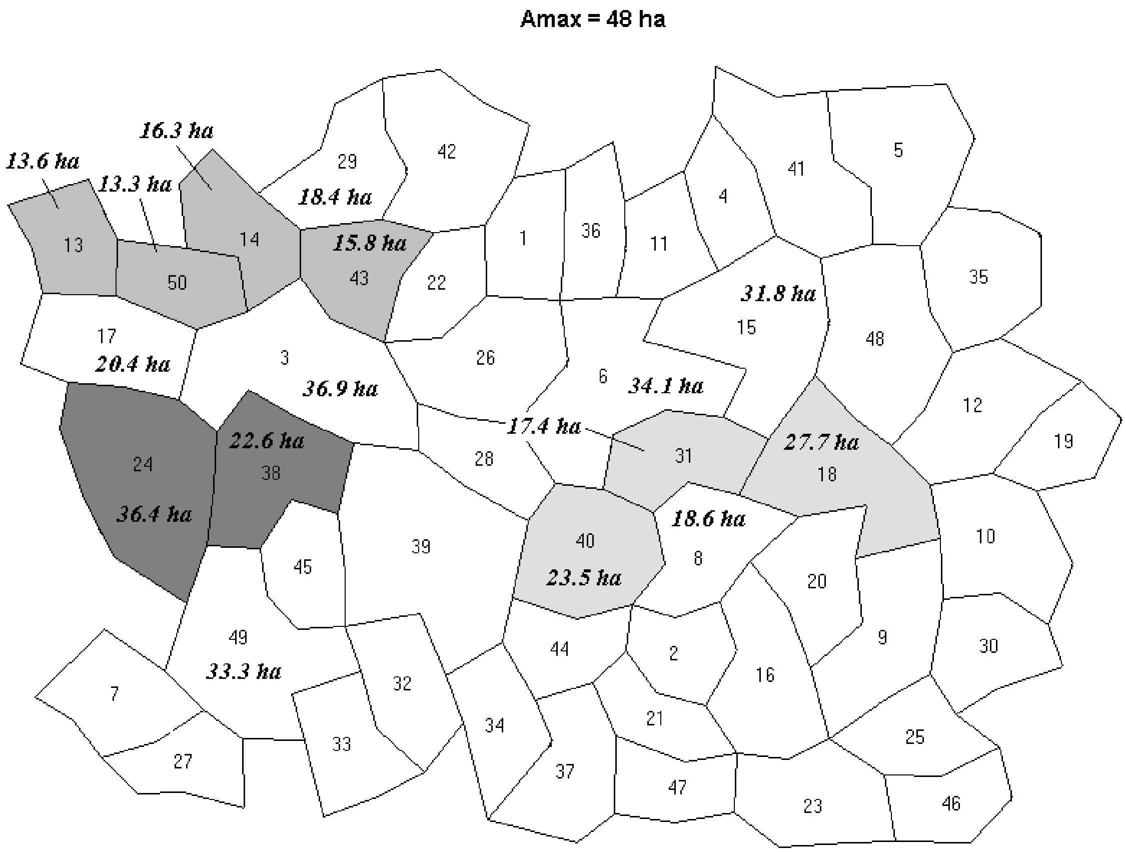

Figure 1: Strengthening 2-, 3-, and 4-Way Covers

patch habitat constraints is that the resulting models are large, complex, and hard to solve. Improving either the structure of the harvest opening size constraints or the mature patch habitat constraints can potentially make these models easier to solve. The focus of this study is to improve the structure of the harvest opening size constraints.

The simplest type of maximum harvest opening size constraints prevents adjacent management units from being harvested within the same time period (McDill and Braze 2000). This case, referred to as the Unit Restriction Model (URM, Murray 1999), assumes that the combined area of any two units in the forest would exceed this maximum area. The Area Restriction Model (ARM, Murray 1999) is more general, allowing groups of contiguous management units to be harvested concurrently as long as their combined area is less than the maximum opening size ($A_{\max}$). Depending on the average area of management units, the maximum harvest opening size, and the age-class distribution of the forest, the ARM formulation might allow for a significantly higher net present value (NPV) of the forest. Unfortunately, formulating and solving forest planning problems with ARM constraints is generally considerably more difficult than formulating and solving such problems with URM constraints. In fact, ARM problems were initially deemed impossible to formulate in a linear model (Murray 1999) and only heuristics were employed to solve them (e.g., Lockwood and Moore 1993, Caro et al. 2003, Richards and Gunn 2003).

McDill et al. (2002) were among the first to develop exact, linear, 0-1 programming formulations of the ARM. One of their two formulations uses constraints that are designed to allow groups of contiguous management units to be harvested as long as their combined area does not exceed the maximum harvest opening limit. McDill et al.(2002) present an algorithm, which they call the Path Algorithm, that recursively enumerates all sets of contiguous management units, set C, whose combined areas just exceed the maximum allowable harvest size. The constraints created this way are similar to cover inequalities in 0-1 knapsack problems, thus we frequently refer to the path constraints as cover inequalities in this paper. The disadvantage of this Path/Cover formulation is that the number of constraints that are required can be very large and it can increase exponentially as the ratio of the average size of a management unit to the maximum allowable harvest opening area decreases. The advantage of the Path/Cover formulation over McDill et al.'s (2002) second formulation, discussed next, is that it does not require the introduction of additional 0-1 decision variables.

McDill et al.'s (2002) second formulation uses separate variables for each possible combination of contiguous management units within the forest whose total area does not exceed the allowable harvest opening size. The authors refer to these combinations as Generalized Management Units (GMUs). For each adjacent pair (or clique) of original management units, a pairwise (or clique) adjacency constraint is written, where the set of decision variables include all of the variables that correspond to the GMUs that contain the original units. Goycoolea et al. (2005) applied maximal clique constraints to GMUs to formulate ARM problems and found that these formulations performed better. They also showed that the maximal clique GMU formulation, which they called as the *Cluster Formulation* is at least as tight or tighter than the Path Formulation and leads to better linear programming relaxations (Goycoolea et al. 2009).

The third exact 0-1 programming formulation of ARM was proposed by Constantino et al. (2008). This approach is very different from the Path/Cover and GMU/Cluster formulations in that it does not rely on a recursive, potentially time consuming *a priori* enumeration of spatial constructs such as minimally infeasible (as in the Path Model) or feasible clusters of management units (as in the GMU Model). The recognition that the number of clearcuts in a forest cannot exceed the number of management units gives rise to the definition of a parsimonious set of *clearcut assignment variables* that represent the decision whether a unit should be assigned to a particular clearcut (also referred to as "bucket" in Goycoolea et al. 2009) or not in a given planning period. In the context of Constantino et al.'s (2008) model, a clearcut or *bucket* may comprise units that are disconnected. Additional constraints are present in the formulation to ensure that the area of these clearcuts never exceeds the maximum opening size and that two or more clearcuts never overlap or are never adjacent. Since the number of assignment variables in this formulation is bounded by $n \times n$, where n is the number of management units in the forest, Constantino et al.'s (2008) model leads to smaller problems than the other two formulations when the maximum harvest opening size is large. Further, substantial reductions in problem size are possible by eliminating those assignments from the model where the area of the minimum-area path between the two management units involved is greater than the maximum harvest opening size. Such assignments can be found very efficiently by the Floyd-Warshall (Roy 1959, Floyd 1962 or Warshall 1962) or other minimum-weight shortest path algorithms. Recent findings evidence that Goycoolea et al.'s (2005) maximal clique constraints provide a tighter approximation of the convex hull of the ARM than the constraints in the Bucket Model (Martins et al. 2011).

Lastly, we mention two other formulations of the ARM, one of which can be viewed as an extension of the Path model. Crowe et al. (2003) appended what they call "ARM clique constraints" to McDill et al.'s (2002) cover inequalities, arguing that the "clique" concept can be applied to ARM models if the total area of a mutually adjacent set of management units exceeds the maximum opening size. Crowe et al. (2003) "clique constraints" are written for each mutually adjacent set of units, where the left-hand-side coefficients are the areas of the units and the right-hand-side is the allowable cut limit. Crowe et al. (2003) found that the appended formulation did not outperform the Path/Cover approach computationally. It can be shown, however, that some of these ARM clique constraints cut off fractional solutions from the feasible set defined by the Path/Cover formulation, and thus they may be used to better approximate the ARM mathematically.

The same can be said about Gunn and Richards' (2005) "stand-centered" constraints that can also be an alternative to or complement McDill et al.'s (2002) cover inequalities. One stand-centered constraint is written for each management unit and period. The constraint prevents the harvest of the unit in the specified period if the combined area of the adjacent units that are scheduled for harvest in that period exceeds the cut limit minus the area of the unit. Gunn and Richards (2005) observe that these constraints do not prevent every possible harvest area violation, but they argue that these violations will be few when the areas of management units are not too small compared with the harvest opening area limit and that those that do occur can be easily detected and "post-fixed" at a relatively small loss in optimality. Although Gunn and Richards' (2005) model is not an exact formulation of the ARM, it is attractive for two reasons. First, the number of stand-centered constraints needed is equal to the number of units in a forest, which is much less than the number of covers that might be needed.

Second, unlike finding McDill et al.'s (2002) covers, generating stand-centered constraints does not require a potentially very time-consuming recursive enumeration.

In this paper, we present a procedure that strengthens the cover inequalities introduced as path constraints by McDill et al. (2002), and show that the strengthened formulation can lead to solution times that are shorter than what is provided by the other models. The contribution is in the procedure itself that leads to a demonstrably tighter formulation of the ARM than the Path/Cover model without introducing additional variables. While the proposed procedure is conceptually similar to the one that have been used for 0-1 knapsack problems in the operations research literature (Wolsey 1998, p.147), the ARM requires sequential coefficient lifting because of the adjacency restrictions that are imposed on the management units. This additional level of complexity leads to a markedly different, and more complex, strengthening algorithm than the one used for 0-1 knapsack inequalities.

In the next section, we formally describe the three existing integer programming formulations of the ARM that we will use as benchmarks to assess the performance of the strengthened model. The structure of the strengthened path/cover constraints will be discussed next, followed by a description of the strengthening algorithm that can be used to automate the process of creating these constraints. The computational efficiency of the strengthened formulation is assessed by formulating and solving sixty hypothetical and seven real harvest scheduling problems in four ways: (1) with the original path/cover inequalities of McDill et al.(2002), (2) with Goycoolea et al.'s (2005) maximal clique-based Cluster Formulation, (3) with Constantino et al.'s (2008) Bucket Formulation, and (4) with the strengthened pathy/cover inequalities proposed in this paper. We find that not only does the strengthened model lead to better solution times in many application instances, but the results also suggest that the original Path Formulation is still competitive relative the Cluster Model. This is a somewhat surprising result given that the performance of the Cluster Model has previously been found to be superior to that of the Path Model (Goycoolea et al. 2009). The paper concludes with a discussion on how the strengthening procedure could potentially be further improved.

2 The benchmark ARM model formulations

2.1 The Path/Cover Model (McDill et al. 2002)

The general structure of the spatially explicit ARM model, where the adjacency constraints are generated by the Path Algorithm, is as follows:

$$Max\,Z = \sum_{m=1}^{M} A_m[c_{m0}x_{m0} + \sum_{t=h_m}^{T} c_{mt}x_{mt}] \qquad (1)$$

Subject to:

$$x_{m0} + \sum_{t=h_m}^{T} x_{mt} \leq 1 \qquad (2)$$

for $m = 1, 2, ..., M$

$$\sum_{m \in M_{ht}} v_{mt} \cdot A_m \cdot x_{mt} - H_t = 0 \qquad (3)$$

for $t = 1, 2, ... T$

$$b_{l,t}H_t - H_{t+1} \leq 0 \qquad (4)$$

for $t = 1, 2, ... T$-1

$$-b_{h,t}H_t + H_{t+1} \leq 0 \qquad (5)$$

for $t = 1, 2, ... T$-1

$$\sum_{j \in C} x_{jt} \leq |C| - 1 \qquad (6)$$

$\forall C \in \mathbb{C}$ and for $t = h_C, ... T$

$$\sum_{m=1}^{M} A_m[(Age_{m0}^T - \overline{Age}^T)x_{m0} + \sum_{t=h_m}^{T} (Age_{mt}^T - \overline{Age}^T)x_{mt}] \geq 0 \qquad (7)$$

$$x_{mt} \in \{0, 1\} \qquad (8)$$

for $m = 1, 2, ..., M$ and $t = h_m, ... T$

$$H_t \geq 0 \qquad (9)$$

for t = 1,... T, where:

h_m = the first period in which management unit m (m is the unit ID) is old enough to be harvested,

x_{mt} = a binary variable whose value is 1 if management unit m is to be harvested in period t for $t = h_m, ... T$; when $t = 0$, the value of the binary variable is 1 if management unit m is not harvested at all during the planning horizon (i.e., x_{m0} represents the "do-nothing" alternative for management unit m),

M = the number of management units in the forest,

T = the number of periods in the planning horizon,

c_{mt} = the discounted net revenue per hectare plus the discounted expected forest value at the end of the planning horizon if management unit m is harvested in period t. If unit m is not cut at all (i.e., $x_{m0} = 1$), then c_{m0} is equal to the discounted expected forest value at the end of the planning horizon.

M_{ht} = the set of management units that are old enough to be harvested in period t,

A_m = the area of management unit m in hectares,

v_{mt} = the volume of sawtimber in m^3/ha harvested from management unit m if it is harvested in period t,

H_t = the total volume of sawtimber in m^3 harvested in period t,

$b_{l,t}$ = a lower bound on decreases in the harvest level between periods t and $t+1$ (where, for example, $b_{l,t} = 1$ would require non-declining harvests and $b_{l,t} = 0.9$ would allow a decrease of up to 10%),

$b_{h,t}$ = an upper bound on increases in the harvest level between periods t and $t+1$ (where $b_{h,t}$ = 1 would allow no increase in the harvest level and $b_{h,t}$ = 1.1 would allow an increase of up to 10%),

C = the set of indexes corresponding to the management units in cover C,

$\mathbb{C}$ = the set of covers that arise from a forest planning problem,

h_C = the first period in which the youngest management unit in cover C is old enough to be harvested,

Age^T_{mt} = the age of management unit m at the end of the planning horizon if it is harvested in period t; and

$\overline{Age}^T$ = the minimum average age of the forest at the end of the planning horizon.

Equation (1) specifies the objective function of the problem, namely to maximize the discounted net revenue from the forest during the planning horizon plus the discounted forest value of each stand at the end of the planning horizon. The first set of constraints (2) consists of logical constraints. They require a management unit to be assigned to at most one prescription, including a do-nothing prescription. Harvest variables (x_{mt}) are only created for periods where the stand is old enough to be harvested (i.e., it is older in that period than the predefined minimum rotation age). The second set of constraints (3) consists of harvest accounting constraints. They sum the harvest volume for each period and assign the resulting value to the (continuous) harvest accounting variables (H_t). Constraint sets (4) and (5) are flow constraints. They limit the rate at which the harvest volume can increase or decrease from one period to the next. Constraint set (6) represents the maximum harvest opening constraints as minimal covers generated by the Path Algorithm. These constraints assume that the exclusion period equals one planning period, i.e., that once a management unit, or group of contiguous units, has been harvested, no adjacent management units can be harvested until at least one period has passed. The structure of these constraints is easy to generalize to alternative exclusion periods which are integer multiples of a planning period (see, for example, Snyder and ReVelle 1997b). Constraint (7) is an ending age constraint. It requires the average age of the forest at the end of the planning horizon to be at least $\overline{Age}^T$ years. These constraints help prevent the model from over-harvesting the forest during the planning horizon and define a minimum criterion for a desirable ending condition of the forest. Constraint (8) identifies the management unit treatment alternative variables as binary. Constraints (9) restrict the harvest volume accounting variables to be positive and continuous.

2.2 The Cluster Model (Goycoolea et al. 2005 – EARM-4) Unlike the Path Model that uses management unit-based variables, the Cluster Model uses cluster variables, $x_{ut}\ \forall u \in G,\ t = h_u, ..., T$ where:

u = indexes clusters of management units, $m \in M$, that form connected sub-graphs within the adjacency graph associated with the set of units in the forest with a condition that $\sum_{j \in u} a_j \leq A_{\max}$. The adjacency graph of a forest planning problem is a set of nodes that represent the management units, and a set of edges that represent the adjacency among the units. Only units sharing a common boundary are considered adjacent.

G = the entire set of clusters that arise from a particular forest; and

h_u = the first period in which the youngest management unit in Cluster u is old enough to be cut.

Cluster variable x_{ut} takes the value of one if Cluster u is to be cut in period t, 0 otherwise. The variable x_{u0} represents the special decision whether Cluster u should be cut during the entire planning horizon: a value of one indicates that it should not and a zero indicates that it should.

The Cluster Formulation requires a set of logical constraints that are slightly different from those of the Path Model (Constraints 2):

$$\sum_{u \in G_m} \left(x_{u0} + \sum_{t=h_u}^{T} x_{ut} \right) \leq 1 \qquad (10)$$

for m = 1, 2, ...,M, where G_m denotes the set of clusters that contain management unit m. Constraint set (10) simply requires that unit m can only be cut or not cut as part of at most one cluster.

To ensure that the clusters that are cut in the same planning period are never adjacent or overlapping, the following inequality is added to the model for each maximal clique of management units and for each time period when the youngest unit in the clique is old enough to be cut. Maximal cliques are sets of mutually adjacent management units that are not strictly subsets of any other cliques:

$$\sum_{n \in K_{jt}} x_{nt} \leq 1 \qquad (11)$$

for all $j \in J$ and $t = h_j$, ..., T, where:

K_{jt} = the set of indexes corresponding to the set of clusters that 1) contain at least one unit in maximal clique j of the original management units and 2) where all the stands comprising the cluster are old enough to be harvested in period t.

h_j = the first period in which the youngest unit in clique j is old enough to be harvested, and

J = the entire set of maximal cliques of management units that exist in the forest.

Constraint set (11) prevents maximum clear-cut size violations. The harvest volume accounting and flow constraints, as well as the minimum average ending age constraint are the same as in the Path Model (inequalities 3-5 and 7), except that the unit variables are replaced with cluster variables and the volume, area and age coefficients refer to the clusters instead of the management units.

2.3 The Bucket Model (Constantino et al. 2008 – ARMSCV-C) To formulate the Bucket Model, we define K as a class of *clearcuts*. Each clearcut is uniquely indexed by a management unit (stand). Thus, $|\text{K}| = M$, where M is the number of units in the forest. Further, the elements of a clearcut $K_i \in \text{K}$ are management units defined by the following function (0-1 program). Function (12)-(15) assigns a set of units, (which can be the empty set) to each clearcut via the use of binary variables x_m^{it} that take the value of 1 if unit m is assigned to clearcut i in period t. The value of this variable is 0 otherwise.

$$Max\, Z = \sum_{m=1}^{M} \sum_{i \in \text{K}} a_m [c_{m0} x_m^{i0} + \sum_{t=h_m}^{T} c_{mt} x_m^{it}] \qquad (12)$$

Subject to:

$$\sum_{t=0, t=h_m}^{T} \sum_{i \in \text{K}} x_m^{it} \leq 1 \qquad (13)$$

for $m = 1, 2, ..., M$

$$\sum_{m=1}^{M} a_m x_m^{it} \leq A_{\max} \qquad (14)$$

for $i \in \text{K}$ and $t = h_m, \ldots T$

$$x_m^{it} \in \{0, 1\} \qquad (15)$$

for $i \in \text{K} : i \leq m$ and $t = h_m, \ldots T$.

Equation (12), the objective function, is equivalent to Equation (1) in the Path Model. It maximizes the discounted net timber revenues from the forest over the planning horizon. Constraint set (13) comprises the logical constraints for the Bucket Model. They allow a management unit to be harvested only once in the planning horizon or not at all. Constraints (14) prevent the formation of any clearcut i in class K whose area exceeds the maximum harvest opening size. Lastly, constraint set (15) defines variables x_m^{it} as binary. We note that the assignment variables are only defined for $i \leq m$ to minimize problem size.

Note that since constraint set (14) does not prevent clearcuts in class K from being adjacent or overlapping, it alone cannot prevent maximum harvest opening size violations. Additional constraints are necessary. To that end, Constantino et al.'s (2008) model introduces a new set of binary variables of form w_Q^{it} that take the value of one whenever a unit in maximal clique $Q \in \mathbb{Q}$ is assigned to clearcut i in period t. As with the GMU/Cluster Model (Section 2.1), set $\mathbb{Q}$, the set of maximal cliques of management units, must be enumerated during the model formulation phase. The following two constraint sets, along with constraints (14) guarantee that the maximum harvest opening size is never exceeded. The contribution of constraint sets (16)-(17) is to ensure that the units in each maximal clique can only belong to at most one clearcut in any given planning period:

$$x_m^{it} \leq w_Q^{it} \qquad (16)$$

for $Q \in \mathbb{Q}$, $m \in Q$, $i \leq m$ and $t = h_m, \ldots T$

$$\sum_{i \in \text{K}} w_Q^{it} \leq 1 \qquad (17)$$

for $Q \in \mathbb{Q}$ and $t = h_m, \ldots T$

$$w_Q^{it} \in \{0, 1\} \qquad (18)$$

for $i \in \text{K}$, $Q \in \mathbb{Q}$ and $t = h_m, \ldots T$.

To account for harvest volumes in each planning period and to ensure a minimum average ending age, we modify constraint set (3) and (7) and add them to the

Bucket Model (19-20). The harvest flow constraints are identical to constraint sets (4-5).

$$\sum_{m\in M_{ht},\, i\in \mathrm{K}} v_{mt}\cdot a_m\cdot\; x_m^{it} - H_t = 0 \qquad (19)$$

for $t = 1, 2, . . . \ T$

$$\sum_{i\in\mathrm{K}}\sum_{m=1}^{M} a_m[(Age_{m0}^T - \overline{Age}^T)x_m^{i0} + \\ + \sum_{t=h_m}^{T}(Age_{mt}^T - \overline{Age}^T)x_m^{it}] \geq 0 \qquad (20)$$

The model defined by (12-19) and (4, 5) is identical to what Constantino et al. (2008) refer to as ARMSCV-C. We add a minimum average ending age constraint (20) to this model to prevent the forest from being over-harvested. Finally, we used a variety of pre-processing techniques, as proposed in Constantino et al. (2008) that can reduce the size of the Bucket Model and improve its computational performance. In the next section, we show how the path/cover inequalities can be strengthened to make the Path Model tighter and potentially easier to solve.

3 The Strengthened Path/Cover Formulation

The Strengthened Path Model is identical to the original Path Model, Constraints (1)-(9) except that Constraints (6) are replaced with a stronger inequality set: set (21) below. The next two sub-sections describe how this new inequality set can be derived from the original path constraints, constraint set (6).

3.1 Strengthening the path/cover inequalities

In order to strengthen McDill et al.'s (2002) path formulation, the structure of the path/cover constraints must be studied first. The minimal path/cover inequalities generated by McDill et al.'s (2002) Path Algorithm are of form $\sum_{j\in C} x_{jt} \leq |C| - 1$, where C is a set of management units that form a connected sub-graph of the underlying adjacency graph, and for which $\sum_{j\in C} a_j > A_{\max}$ holds, but $\sum_{j\in C\setminus\{l\}} a_j \leq A_{\max}$ for any $l \in C$ such that set $C\setminus\{l\}$ is still a connected sub-graph.

Set C can be called a "path," as in McDill et al.(2002), or, using the analogy with the cover inequalities that arise in 0-1 Knapsack problems, it can be called a "cover" (Wolsey 1998). These covers are minimal connected sub-graphs because if any one unit is excluded from C, the total area of the remaining management units will be less than the harvest limit. For example in Figure 1, the set of management units {13, 14, 43, 50} forms a minimal cover given an $A_{\max}$ of 48 ha. Management unit IDs are listed for each polygon, along with the area of the units in bold that are relevant to the discussion. As an example, unit 50 is 13.3 ha in size. Throughout the rest of this paper, it is assumed that each management unit in the forest has an area less than or equal to the allowable contiguous cut limit. Thus, $|C| \geq 2$, for any $C \in \mathbb{C}$ with $\mathbb{C}$ being the complete set of all possible minimal covers that arise from the forest planning problem.

To establish the notation that is necessary for the strengthening procedure, we define the feasible region of the ARM based on the Path Formulation but without the logical, harvest flow and ending age constraints (2)-(5) and (7): $P = \{x^t \in \{0,1\}^n : \sum_{i\in C} x_{it} \leq |C| - 1,\ \forall \mathrm{C} \in \mathbb{C},\ t = h_C, ..., T\}$, where n is the number of units in the forest. For every set of management units A, let $N(A)$ represent the set of all management units adjacent, but not belonging, to A. Finally, let function $\pi(s,t,C) = \max\left\{\sum_{j\in N(s)\bigcap C} x_{jt} :\ x^t \in P \text{ and } \mathrm{x_{st}} = 1\right\}$ define the maximum number of management units in Cover C that are adjacent to unit s and can be cut concurrently with unit s in period t. Note that while function $\pi(s,t,C)$ is an integer program in itself, it is trivial to solve by querying set C , which has already been found via Algorithm I (Goycoolea et al. 2009). The query can be instructed to return the maximum cardinality of minimal covers that (1) comprise units exclusively from set$\{s \bigcup \{N(s) \bigcap C\}\}$ that can be cut in period t, and (2) contain unit s. The value of function $\pi(s,t,C)$ is equal to the maximum cardinality returned by the query minus 2. Then, $\alpha_{st}^* = (|N(s) \bigcap C| - \pi(s,t,C) - 1)$ if $h_s \leq t$, 0 otherwise.

Proposition 1: Consider a minimal cover C and unit $s \in N(C) : h_s \leq t$. Define $\alpha_{st}^* = (|N(s) \bigcap C| - \pi(s,t,C) - 1)$ as the coefficient of variable x_{st}. Then, for all $\alpha_{st} \leq \alpha_{st}^*$:

$$\sum_{j\in C} x_{jt} + \alpha_{st}x_{st} \leq |C| - 1 \qquad (21)$$

is valid for P.

Proof: Consider $x^t \in P$. If $x_{st} = 0$, then the inequality holds by the definition of minimal cover C. If $x_{st} = 1$, then

$$\begin{aligned}\sum_{j\in C} x_{jt} + \alpha_{st}x_{st} &= \sum_{j\in C\setminus N(s)} x_{jt} + \sum_{j\in N(s)\cap C} x_{jt} + \alpha_{st} \\ &\leq |C\setminus N(s)| + \sum_{j\in N(s)\cap C} x_{jt} + \alpha_{st}^* \\ &\leq |C\setminus N(s)| + \pi(s,t,C) + \alpha_{st}^* \\ &= |C| - 1 \qquad \blacksquare\end{aligned}$$

After the terminology established by Wolsey (1998) and others for 0-1 knapsack polytopes, we call

$\sum_{j \in C} x_{jt} + \alpha_{st} x_{st} \leq |C| - 1$ for any $1 \leq \alpha_{st} \leq \alpha^*_{st}$ an extended cover inequality and the associated set, $E(C) = C \cup \{s\}$ an extended cover.

To illustrate the use of Proposition 1, consider the 4-way cover, $C = \{13, 14, 43, 50\}$, in the 50-stand hypothetical forest planning problem shown in Figure 1. Taking $s = \{3\}$, we have that inequality $x_{13,t} + x_{14,t} + x_{43,t} + x_{50,t} + \alpha_{3,t} x_{3,t} \leq 3$, where $\alpha_{3,t} \leq 2$, is valid for P. Also, note that neither $s = \{17\}$, nor $s = \{29\}$ yield stronger inequalities than $x_{13,t} + x_{14,t} + x_{43,t} + x_{50,t} \leq 3$.

To see that Proposition 1 does not always lead to the strongest possible inequalities, consider a forest that comprises only five units: unit 1, 2, 3, 4 and s. Assume that $a_{st} = 2$, $a_{1t} = a_{2t} = a_{3t} = a_{4t} = 1$, and $A_{\max} = 3$. Now suppose also that $N(1) = \{2, 3, 4\}$, $N(2) = \{1, 3, s\}$, $N(3) = \{1, 2, 4, s\}$, $N(4) = \{1, 3, s\}$ and $N(s) = \{2, 3, 4\}$. Clearly, $C = \{1, 2, 3, 4\}$ is a cover and Proposition 1 leads to $E(C) = C \cup \{s\}$ with $\alpha^*_{st} = 1$. However, $E(C)$ with $\alpha_{st} = 2$ is also valid and is stronger.

Now, consider a situation where, for a given minimal cover C, there exist two or more management units, $s \in N(C)$, for which $\alpha^*_{st} \geq 1$. Let Q_{Ct} denote this unique set of management units, which we will call the "cover extension set" for C in time t. The question is, when $|Q_{Ct}| \geq 2$, which subset or subsets of Q_{Ct} can be added, and with what coefficients to minimal cover C so that the resulting constraint(s) would be valid in P. To illustrate this situation, consider the minimal cover $C = \{18, 31, 40\}$ in Figure 1. Using Proposition 1, we find that $Q_{Ct}=\{6, 8, 15\}$. While extended covers $\{18, 31, 40, 6, 15\}$ and $\{18, 31, 40, 8\}$ both lead to valid inequalities, $\{18, 31, 40, 6, 8, 15\}$ does not.

We use the term "compatible" to describe subsets of Q_{Ct} that can be included together in a single extended cover constraint. The issue of compatibility is complicated by the fact that in some cases the coefficients of the units in the cover extension set can be lifted to $\alpha^*_{st}>1$ when they are included singly in an extended cover constraint, but these units may only be compatible with other units in the cover extension set when their coefficients are not lifted (i.e., $\alpha_{st}= 1$) or when the coefficients are lifted but not all the way to α^*_{st} (e.g., $\alpha_{st}= 2 < \alpha^*_{st}=3$).

Thus, for a set of coefficient values, $\mathrm{A}_{Ct} = \{\alpha_{st} : \alpha_{st} \in \mathbb{R}^+,\ \alpha_{st} \leq \alpha^*_{st}\ \mathit{and}\ s \in Q_{Ct}\}$ for minimal cover C, we define the Compatibility Problem:

$$P_{\mathrm{A},C,t} = \max \left\{ \sum_{s \in Q_{Ct}} \alpha_{st} x_{st} + \right. \left. + \sum_{j \in C} x_{jt} : \forall \alpha_{st} \in \mathrm{A}_{Ct},\ x^t \in P \right\}.$$

$P_{\mathrm{A},C,t}$ is a unique integer program that is trivial to solve because the number of decision variables in the objective function is very small. If $P_{\mathrm{A},C,t} \leq |C| - 1$, then, obviously, $\sum_{s \in Q_{Ct}} \alpha_{st} x_{st} + \sum_{j \in C} x_{jt} \leq |C| - 1$ is valid in P. We will refer to the evaluation of $P_{\mathrm{A},C,t}$ as the Compatibility Test.

3.2 The Strengthening Algorithm

The Strengthening Algorithm generates all of the non-dominated extended cover inequalities that can be developed from the initial set of minimal covers. The goal is to tightly approximate the convex hull of the Path/Cover Model (*conv(X)*) described in Section 2.1. The flowchart in Figure 2 illustrates each step of the algorithm.

The Strengthening Algorithm starts by selecting a cover C from the complete set of minimal covers $\mathbb{C}$ (Step [1] in Figure 2), generated by the Path Algorithm (McDill et al. 2002). The set of management units that are adjacent to at least two units in C, but not belonging to C, is identified next [Step 2]. Let S ($S \subseteq N(C)$) denote this set (note: Proposition 1 implies that units that are adjacent to only one unit in C cannot have an α^*_{st}= 1). A management unit s is selected from set S [Step 3] and the maximum value of its coefficient (α^*_{st}) is calculated for period $t = 1$ [Step 4] using Proposition 1 [Step 5]. If $\alpha^*_{st} \geq 1$, then unit s enters the set Q_{Ct} [6]. Once all of the units in S have been processed [7], the cardinality of Q_{Ct} is evaluated. If $|Q_{Ct}| = 0$, the minimal cover C cannot be strengthened [8], and the cover is discarded from $\mathbb{C}$ and a new minimal cover is selected for evaluation [Step 25]. If $|Q_{Ct}| = 1$, i.e., there is only one unit, say unit m, that can be included in the cover, a new, valid extended cover inequality of form $\sum_{j \in C} x_{jt} + \alpha^*_{mt} x_{mt} \leq |C| - 1$ is built and added to the integer programming problem [9]. Then, the algorithm checks if the value of t is equal to the last period T [24]. If it is not, it moves to the next planning period [3] and selects another candidate unit for cover extension (unit s) and Proposition 1 is used again to calculate α^*_{st}. If the value of t is already equal to T, then Cover C is discarded from $\mathbb{C}$ and a new minimal cover is selected Steps [25] and [1]. If in Step [9], $|Q_{Ct}| > 1$, the *Compatibility Algorithm* is launched (Figure 2) to determine which subsets of cover extension set Q_{Ct} can be used to form valid extended cover inequalities, and with what maximum coefficients.

As the first step of the Compatibility Algorithm, the coefficients of all management units in Q_{Ct} are lifted to the maximum values determined by Proposition 1 [10] (i.e., for $\forall s \in Q_{Ct}, \alpha_{st} := \alpha^*_{st}$). Let A_{Ct} denote this coefficient set. Integer program $P_{\mathrm{A},C,t}$ is formulated using A_{Ct} and is evaluated using the Compatibility Test [11]. If Coefficient Set A_{Ct} passes the test, i.e., if the solution to $P_{\mathrm{A},C,t}$ is less than or equal to

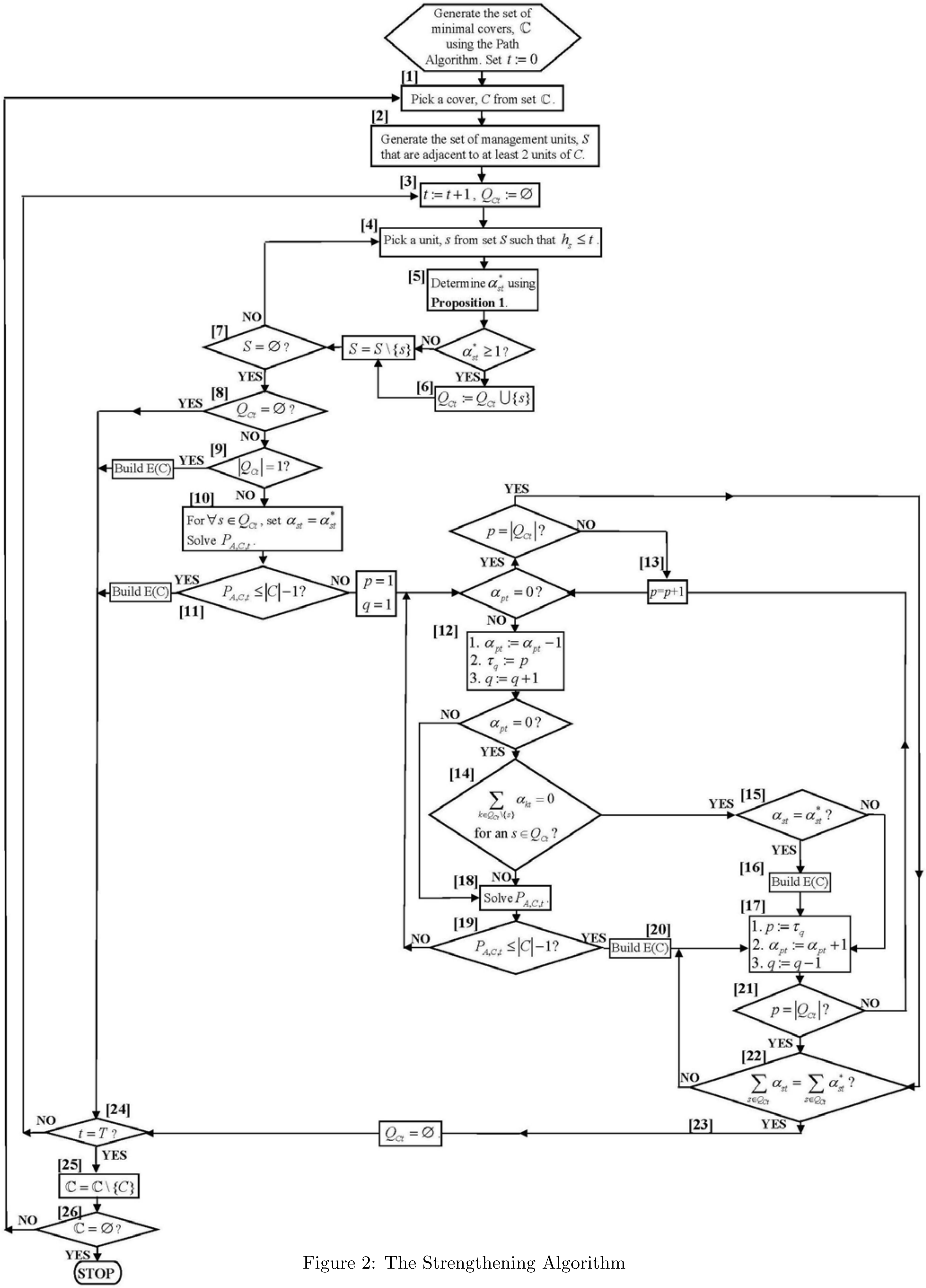

Figure 2: The Strengthening Algorithm

$|C|-1$ then a new, valid extended cover inequality of form $\sum_{j\in C} x_{jt} + \sum_{s\in Q_{Ct}} \alpha^*_{st} x_{st} \le |C|-1$ is built and added to the integer programming problem. If Coefficient Set A_{Ct} does not pass the test, a recursive, branching process takes place.

Coefficient Set A_{Ct}, as constructed in the previous step, forms the root node of the subsequent branching process. Starting from the root node, the algorithm systematically reduces the coefficient values of the units in Q_{Ct} and determines, using the Compatibility Test, which coefficient sets would lead to valid extended inequalities. Each node in the branching process represents a coefficient set. If the extended cover inequality at a node is not valid, new coefficient sets (i.e., new nodes) are generated by reducing each coefficient by one, one at a time. The algorithm terminates once all the nodes are explored or if the remaining nodes represent coefficient sets each containing only one non-zero coefficient (e.g., {2, 0, 0}).

In the first step of the branching process, a unit is selected from Q_{Ct} and the value of its coefficient is reduced by one ([12] in Figure 2). If the coefficient of the selected unit is already zero, which can happen if the algorithm loops back to this unit repeatedly, then another unit is selected from Q_{Ct} [13]. Every time a coefficient of a unit is reduced, the unit ID is recorded in an array indexed by τ_q [12], where q counts the unit IDs. The structure of this array is explained later in the text.

If the reduced coefficient of the unit is zero then the algorithm tests if there is another unit in Q_{Ct}, say unit s, for which $\alpha_{st} = \alpha^*_{st}$ while the coefficients of the rest of the units are zero [14-15]. If this is the case then a new, extended cover inequality of form $\sum_{j\in C} x_{jt} + \alpha^*_{st} x_{st} \le |C|-1$ is built and added to the integer programming problem [16]. This inequality is valid by Proposition 1. The branch that leads to the coefficient set that gives rise to this inequality is fathomed [17]. If $\alpha_{st} < \alpha^*_{st}$ while the coefficients of the rest of the units are zero, then this coefficient set is ignored and the branch is fathomed [17]. This coefficient set can be ignored because the inequality that would result from this set, $\sum_{j\in C} x_{jt} + \alpha_{st} x_{st} \le |C|-1$, is dominated by a valid inequality: $\sum_{j\in C} x_{jt} + \alpha^*_{st} x_{st} \le |C|-1$ (see Proposition 1 and step [5] in Figure 2).

If the reduced coefficient of the unit is not zero, or if it is zero but there are at least two other units in the cover extension set with non-zero coefficients, then $P_{A,C,t}$ is formulated and solved [18]. If $P_{A,C,t} \le |C|-1$ [19], then a new extended cover inequality of form $\sum_{s\in Q_{Ct}} \alpha_{st} x_{st} + \sum_{j\in C} x_{jt} \le |C|-1$ (where $\alpha_{st} \in A_{Ct}$) is built and added to the integer program [20]. The branch that lead to this constraint is fathomed [17]. If $P_{A,C,t} > |C|-1$ then either the coefficient of the unit is further reduced by one [12] or a new unit from set Q_{Ct} is selected for coefficient reduction [13]. The process starts all over again.

Now, going back to step [17] in Figure 2; each time a branch is fathomed, the algorithm backtracks to the parent node. In other words, the coefficient that has been reduced by one to give rise to the branch that was fathomed is recovered (steps 1 and 2 within [17]). Next, the algorithm checks if all the branches from the parent node have been evaluated [21]. If not, a new unit is selected for coefficient-reduction and the process loops back to step [12]. Otherwise, it is tested if the current node is the root node [22]. If it is, no more extended cover inequalities can be derived from minimal cover C [23]. If the current node is not the root node, further "backtracking" takes place in the branching tree. In other words, if no more nodes can be derived and tested from a given node, the algorithm reverts to the parent node using a "backtracking" array whose elements are indexed by τ_q [17]. Array τ_q keeps track of the sequence of coefficient reductions so that the algorithm can loop back to the parent nodes once all branches below that node have been explored. For example, an array of $\{\tau_1 =12,\ \tau_2 =23,\ \tau_3 =54,\ \tau_4 =54\}$ represents a branching path when the coefficient of unit 12 was reduced by one first, then the coefficient of unit 23 was reduced by one and, finally, the coefficient of unit 54 was reduced by one, twice. In Figure 2., index p denotes the unit IDs in set Q_{Ct}.

Figure 3 illustrates a "branching tree" that can be derived from the following coefficient set: $A_{Ct} = \{\alpha^*_{1t} = 2,\ \alpha^*_{2t} = 2,\ \alpha^*_{3t} = 1\}$. If the extended cover inequality that can be derived from this "root" node is valid then it is added to the integer program; no branching/coefficient-reduction is needed. Otherwise, three new nodes are generated by reducing the coefficients of each unit by one, one at a time: {1,2,1}, {2,1,1} and {2,2,0}. Compatibility is then tested at each of these nodes and further branching/coefficient-reduction is performed as needed. As the coefficient reduction is done sequentially from unit 1 to unit 3, as in Figure 3, some redundant nodes might arise that need to be eliminated. For example, branching on {2,2,0} would lead to nodes that had already been evaluated earlier in the process.

After the Compatibility Algorithm is complete, another minimal cover is selected from C and the strengthening process starts all over again [1]. The Strengthening Algorithm terminates when no more minimal covers remain in $\mathbb{C}$ [26].

The Strengthening Algorithm can generate a number of redundant (dominated) constraints. As an example, the extended cover constraint that can be derived from terminal node {0,2,0} is dominated by the constraint that can be derived from node {1,2,0}. We leave

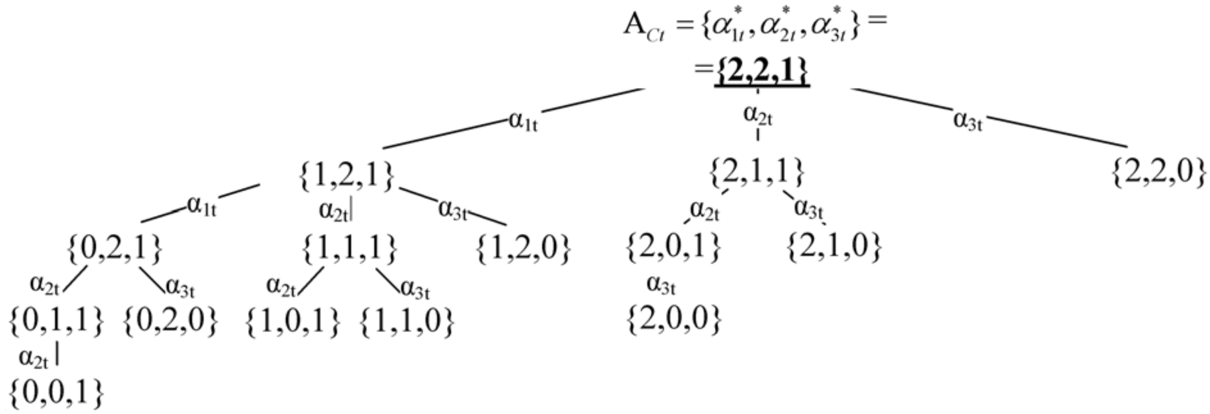

Figure 3: An example of the coefficient reduction procedure.

eliminating these redundancies to the solver's preprocessing algorithms.

Finally, observe that if the Path Algorithm and the strengthening procedure are applied to a URM problem, the resulting extended covers would be maximal cliques. As the combined area of any two adjacent units in the URM exceeds the cut limit, the Path Algorithm would yield only 2-way minimal covers; i.e., pairwise adjacency constraints. Now, for any minimal cover C (i.e., any pair of adjacent units) in a URM, a unit that is a member of the cover extension set Q_{Ct} must be adjacent to both units in cover C (Proposition 1). Additionally, any two or more units in Q_{Ct} can only be compatible, or, equivalently, can only be included jointly in the extended cover, if they are mutually adjacent. If not, they could be cut simultaneously, which would make the extended cover constraint invalid since in a URM the right-hand-sides of the original pair-wise cover constraints are always 1. Thus, for URM problems, all the units in extended covers formed by compatible subsets of Q_{Ct} and the units in the original minimal cover must be mutually adjacent. In other words, they form cliques. Moreover, if the Strengthening Algorithm is set to eliminate all dominated constraints (we left this procedure to the solver's built-in preprocessing routines), the remaining cliques will be maximal. This is an important observation for three reasons. First, it shows that the extended cover inequalities proposed here for the ARM generalize the concept of the maximal clique constraints in URM problems. Second, it implies that the Strengthening Algorithm could potentially be used to generate maximal clique constraints for URM problems, although existing algorithms for this special case likely are more efficient. Third, as the maximal clique formulation has been identified as a tight formulation of the maximum weight stable set problem, it is reasonable to expect that the proposed Strengthening Algorithm would yield a computationally advantageous formulation of the ARM as well.

4 Test problems

The computational efficiency of the strengthened covers was tested on sixty hypothetical and seven real forest planning problems. Thirty of the hypothetical forests had 300 units and thirty had 500 units. The real forests consisted of 32, 71, 89, 90, 1,019, 1,363 and 5,224 units. The hypothetical problems had one forest type and one site class, while the real problems had one, four, five or six forest types and one, two, three or four site classes. Forests in different forest types or site classes exhibit different growth and yield patterns. The initial age-class distribution of the hypothetical forests mimics a typical Pennsylvania hardwood forest. The hypothetical problems were generated in batches using a program called MakeLand (McDill and Braze 2000). MakeLand was instructed to randomly assign age-classes to the polygons of each randomly generated forest map in such a way so that the overall age-class distribution would approximate the age-class distribution of an average Pennsylvania hardwood forest. This random age-class assignment was done three times for each of twenty maps resulting in the thirty 300-stand and thirty 500-stand problems. Neighborhood adjacency (vertex degree) was varied by changing the initial number of points that MakeLand was instructed to use to construct the polygons. The age-classes and yields of each unit in the real problems were based on on-site measurements.

The planning horizon was 60 years for the hypothetical models, and 25, 40 or 50 years for the real problems. The length of the planning periods was 10 years for each problem except El Dorado and Shulkell where it was 5

Table 1: Test problem characteristics

	Problem IDs	Management unit size distribution (ha)									Area	Unit size			Planning	Vertex	Forest	Site
		0-5	6-10	11-15	16-20	21-25	26-30	31-35	36-50	Total	(ha)	Min	Max	Mean	periods	degree	types	classes
Real problems	NBCL5	2,833	1,577	623	211	0	0	0	0	5,244	34,739	0.99	20.23	6.65	4X10yrs	2.87	6	1
	El Dorado	107	421	267	183	134	94	88	69	1,363	21,147	4.05	47.09	15.52	5X5yrs	5.30	1	1
	Shulkell	299	377	188	67	49	16	17	6	1,019	9,443	0.31	39.33	9.27	5X5yrs	4.05	6	6
	Kittaning4	1	3	4	13	5	6	0	0	32	588	4.02	29.32	18.38		3.27	4	2
	FivePoints	0	15	19	10	26	14	6	0	90	1,673	5.80	31.75	18.58	5X10yrs	3.71	5	4
	PhyllisLeeper	6	3	15	30	21	13	1	0	89	1,597	1.25	30.46	17.95		3.19	5	3
	BearTown	0	7	11	20	19	13	1	0	71	1,349	5.96	30.81	19.00		2.90	5	3
300-unit hypothetical problems	75-77	0	147	80	38	20	9	4	2		3,600	5.39	38.25	12.00		4.63		
	81-83	0	135	101	35	17	9	2	1		3,600	5.61	38.84	12.00		5.03		
	87-89	0	132	108	35	11	11	3	0		3,600	5.78	32.56	12.00		4.95		
	90-92	0	143	79	46	20	6	6	0		3,600	5.20	35.00	12.00		4.87		
	93-95	0	130	101	43	17	7	2	0		3,600	5.60	33.86	12.00		4.93		
	96-98	0	133	98	43	13	12	1	0	300	3,600	5.95	31.27	12.00	6X10yrs	5.00	1	1
	99-101	0	140	85	48	18	5	3	1		3,600	5.86	35.51	12.00		4.99		
	102-104	0	141	85	38	24	5	4	3		3,600	5.15	38.89	12.00		4.69		
	189-191	0	143	104	31	15	3	1	3		3,480	5.59	38.56	11.60		5.06		
	192-194	0	156	84	37	15	5	2	1		3,480	5.91	39.29	11.60		5.25		
500-unit hypothetical problems	108-110	0	233	170	45	34	12	6	0		6,000	5.56	34.98	12.00		4.94		
	111-113	0	241	151	72	21	4	9	2		5,725	5.15	39.97	11.45		4.79		
	120-122	0	189	161	82	33	22	9	4		6,750	6.93	39.79	13.50		5.29		
	135-137	0	295	122	58	13	7	2	3		5,300	5.40	39.31	10.60		5.28		
	141-143	0	242	164	56	19	9	10	0		5,800	5.82	34.89	11.60		5.36		
	144-146	0	256	142	48	33	15	5	1	500	5,800	5.70	35.67	11.60	6X10yrs	5.44	1	1
	150-152	0	299	131	39	20	5	3	3		5,300	5.43	39.87	10.60		5.50		
	153-155	0	280	146	55	11	5	2	1		5,300	5.37	35.20	10.60		5.47		
	159-161	31	270	126	53	14	4	1	1		5,000	4.78	38.73	10.00		5.46		
	168-170	0	209	150	88	29	14	9	1		6,300	6.35	36.34	12.60		5.46		

Table 2: Test problem formulation characteristics: cover size distribution.

	Problem IDs	Amax (ha)	Cardinality distribution of covers/paths																Total
			17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	
Real problems	NBCL5,	21.00	0	0	0	0	0	0	0	0	0	0	62	463	1,867	4,148	3,749	1,566	11,855
	Canada	30.00	0	0	0	0	0	0	0	1	193	990	3,328	8,151	11,058	8,413	3,021	163	35,318
		40.00	0	0	0	0	0	26	528	2,620	9,051	27,885	41,540	34,432	20,893	6,554	537	0	144,066
	El Dorado,	48.56	0	0	0	0	0	0	0	0	0	3	536	3,476	6,626	6,205	3,127	657	20,630
	California	60.70	0	0	0	0	0	0	0	0	156	3,424	13,335	20,240	17,543	9,859	2,966	193	67,716
		72.84	0	0	0	0	0	0	36	1,958	18,700	47,749	65,734	54,688	31,672	10,609	1,971	18	233,135
	Shulkell,	40.00	0	0	0	0	5	44	105	183	290	790	1,674	3,014	3,603	2,042	845	378	12,973
	Nova Scotia	60.00	23	223	755	1,626	2,747	2,782	5,328	12,870	21,376	26,141	22,532	13,902	6,798	2,066	394	181	119,734
	Kittaning4		0	0	0	0	0	0	0	0	0	0	0	0	0	4	63	1	68
	FivePoints	50.00	0	0	0	0	0	0	0	0	0	0	0	0	4	41	188	28	261
	PhyllisLeeper		0	0	0	0	0	0	0	0	0	0	0	0	4	17	201	8	230
	BearTown		0	0	0	0	0	0	0	0	0	0	0	0	0	14	123	8	145
300-unit hypothetical problems	75-77		0	0	0	0	0	0	0	0	0	0	34	496	1,055	1,414	704	64	3,767
	81-83		0	0	0	0	0	0	0	0	0	0	0	172	1,126	2,228	837	49	4,412
	87-89		0	0	0	0	0	0	0	0	0	0	0	625	1,191	2,458	747	47	5,068
	90-92		0	0	0	0	0	0	0	0	0	0	0	206	1,067	1,891	650	68	3,882
	93-95		0	0	0	0	0	0	0	0	0	0	0	64	1,623	2,292	737	46	4,762
	96-98		0	0	0	0	0	0	0	0	0	0	0	208	1,393	2,193	829	45	4,668
	99-101		0	0	0	0	0	0	0	0	0	0	0	99	1,379	2,512	727	47	4,764
	102-104		0	0	0	0	0	0	0	0	0	0	12	45	767	1,694	656	71	3,245
	189-191		0	0	0	0	0	0	0	0	0	0	0	175	2,355	2,890	627	45	6,092
	192-194		0	0	0	0	0	0	0	0	0	0	0	408	3,392	2,611	693	52	7,156
		40.00																	
500-unit hypothetical problems	108-110		0	0	0	0	0	0	0	0	0	0	0	120	2,113	3,395	1,297	93	7,018
	111-113		0	0	0	0	0	0	0	0	0	0	16	340	2,686	3,363	984	92	7,481
	120-122		0	0	0	0	0	0	0	0	0	0	0	7	603	3,398	1,471	199	5,678
	135-137		0	0	0	0	0	0	0	0	0	0	227	3,245	7,019	4,782	1,087	58	16,418
	141-143		0	0	0	0	0	0	0	0	0	0	0	513	4,033	4,986	1,169	101	10,802
	144-146		0	0	0	0	0	0	0	0	0	0	2	926	4,564	4,729	1,423	79	11,723
	150-152		0	0	0	0	0	0	0	0	0	0	1	3,299	10,800	5,633	819	71	20,623
	153-155		0	0	0	0	0	0	0	0	0	0	33	3,448	10,482	6,156	920	41	21,080
	159-161		0	0	0	0	0	0	0	0	0	13	1,371	7,076	11,369	5,785	818	34	26,466
	168-170		0	0	0	0	0	0	0	0	0	0	0	53	1,683	3,717	1,807	117	7,377

years. The minimum rotation age was 60 for the hypothetical, 80 for the four small real problems, and it ranged from 20 to 100 years for NBCL5, depending on the forest type. No minimum rotation age was specified for El Dorado and Shulkell. The optimal rotation age, based on maximizing the land expectation value (LEV), was 80 years for the hypothetical, 50 years for the small real problems and 90 years for NBCL5. The possible prescriptions were to cut the management unit in period 1, 2, 3, 4, 5, 6 (in the hypothetical forests) or not at all. A maximum harvest opening size of 40 and 50 ha was imposed on the hypothetical and the real forests, respectively. In NBCL5, the maximum opening size was set to three different levels: 21, 30 and 40 ha. In El Dorado, 48.56, 60.70 and 72.84 ha was used and in Shulkell 40 and 60 ha was applied. All units were smaller than the maximum harvest opening size. In the case of Kittaning4, FivePoints, PhyllisLeeper and BearTown, units greater than 50 ha were delineated into smaller units by a Bureau of Forestry employee using contour lines, roads, trails, streams and shape. In NBCL5 and Shulkell, units greater than 21 and 40 ha, respectively, were excluded as we had no site-specific knowledge to make meaningful delineations. In addition, we excluded those units from NBCL5 that had no yield information. The average age of the forests at the end of the planning horizon was limited to be at least half of the minimum rotation age.

Table 1 summarizes the spatial characteristics of each real problem, and each hypothetical problem batch. In addition to the minimum, maximum and mean unit sizes, the unit size distribution, the total forest area, as well as the vertex degrees and the number of forest types, site classes and planning periods are also listed to give the reader an idea of how simple, or complex, the test problems were.

Each problem was formulated in four different ways, using (1) Goycoolea et al.'s (2005) maximal clique-based Cluster Model, (2) McDill et al.'s (2002) path/cover constraints (a.k.a. the Cell Model), (3) Constantino et al.'s (2008) Bucket Model, and (4) the strengthened covers introduced in this paper. As for the path/cover and the cluster enumerations, we used McDill et al.'s (2002) Path and cluster algorithms for each problem instance, except for NBCL5, El Dorado and Shulkell where we used Goycoolea et al.'s (2009) modified, more efficient version: "Algorithm I". Here, unlike in McDill et al.'s (2002), the clusters and paths/covers are generated simultaneously within the same algorithm. The characteristics of the resulting formulations; the path/cover-size distribution, the number of maximal cliques, feasible clusters, 0-1 variables, and the number of adjacency constraints; are summarized in Table 2-3.

All pre-processing and model formulation tasks were automated using Java and IBM-ILOG CPLEX v. 12.1 Concert Technology (2009) (4-thread, 64-bit, released in 2009) on a Power Edge 2950 server that had four Intel Xeon 5160 central processing units at 3.00Gz frequency and 16GB of random access memory. The operating system was MS Windows Server 2003 R2, Standard x64 Edition with Service Pack 2 (2003). The strengthening procedure was implemented in Visual Basic.Net (2005). Every problem instance was solved on the Power Edge 2950 server with CPLEX 12.1. The relative MIP gap tolerance parameter (optimality gap) was set to $5.e-04$ (0.05%), and the working memory limit was set at 1 GB. Default CPLEX settings were used for all other parameters. CPLEX was instructed to terminate if the solution time for a given problem reached 6 hours. We compared the solution times needed to find the first integer feasible solution within the predefined 0.05% optimality gap. If the desired optimality gap was not reached within 6 hours, than the achieved gaps were used as the bases of comparison (Table 3). To better understand the results regarding the hypothetical problems, we constructed a series of 95% confidence intervals on the mean solution times for each of the four methods (Fig. 4.).

5 Results and discussion

Table 4 reports the solution times and optimality gaps for both the hypothetical and the real problems. For BearTown and PhyllisLeeper, the solver was not able to reach the target 0.05% optimality gap within 6 hours of runtime. In these cases, we report only the achieved optimality gaps: with 0.1153% and 0.0507%, the Strengthened Path performed the best in these two particularly hard problems. Other than these two problems, it was NBCL5 at the 30 ha maximum harvest opening size and El Dorado that that could not be solved to the target gap within 6 hours of runtime. While the Path, the Strengthened Path and the Cluster models all solved these problems to the desired gap, the Bucket Model could only achieve a 0.07% gap for NBCL5 and a 0.08, 0.14 and 0.56% gap for El Dorado at 48.56, 60.7 and 72.84 ha opening sizes, respectively. Overall, in 5 out of the 12 real problem instances, the proposed model was the most successful, preceded by the original Path Formulation that came out ahead 6 times and followed by the Cluster Formulation that lead to the shortest solution time only once.

Of the 60 hypothetical problems, 23 times the Strengthened Model, 21 times the Path Model, 9 times the Bucket and 2 times the Cluster Model reached the target gap within the least amount of time (Table 4). While on average, the Strengthened Model was able to reach the target gap for hypothetical problems faster than the other models, this advantage was statistically significant at the p=0.05 level only in comparison with

Table 3: Test problem formulation characteristics: problem size.

	Problem IDs	Amax	Number of			No. of 0-1 variables			No. of ARM constraints			
		(ha)	max. cliques	clusters	covers	CLUSTER	PATH	BUCKET	CLUSTER	PATH	ST PATH	BUCKET
Real problems	NBCL5,	21.00	4,099	16,701	11,855	109,630	23,422	75,096	15,397	32,691	35,191	164,920
	Canada	30.00		62,368	35,318	337,965		126,431	15,495	88,248	89,716	277,596
		40.00		266,860	144,066	1,360,425		187,363	15,507	326,796	327,547	407,233
	El Dorado,	48.56	2,033	20,047	20,630	128,466	8,184	38,032	9,209	54,024	99,880	187,059
	California	60.70		69,638	67,716	426,012		55,780	9,230	164,401	337,645	274,575
		72.84		249,999	233,135	1,508,178		77,413	9,230	521,154	1,165,635	379,145
	Shulkell,	40.00	1,092	24,796	12,973	155,016	6,240	27,361	5,368	50,562	62,930	97,470
	Nova Scotia	60.00		286,701	119,734	1,726,446		65,827	5,370	425,760	594,935	222,448
	Kittaning4		24	66	68	594	150	420	101	165	181	954
	FivePoints	50.00	87	228	261	1,914	390	1,260	324	653	725	3,626
	PhyllisLeeper		88	199	230	1,734	509	1,452	440	951	1,012	3,977
	BearTown		66	125	145	1,182	411	1,102	325	661	666	2,679
300-unit hypothetical problems	75-77		382	3,175	3,767	13,159	1,832	13,074	2,250	12,160	12,940	13,074
	81-83		433	3,136	4,412	13,237	1,823	13,220	2,578	13,954	15,329	13,220
	87-89		411	3,509	5,068	15,748	1,829	13,923	2,435	15,938	17,438	13,923
	90-92		399	3,156	3,882	12,960	1,828	13,122	2,365	11,982	12,932	13,122
	93-95		395	3,321	4,762	14,473	1,832	14,097	2,342	15,909	16,843	14,097
	96-98		426	3,266	4,668	14,685	1,838	13,778	2,524	16,540	17,862	13,778
	99-101		419	3,278	4,764	14,488	1,834	14,179	2,487	16,283	17,384	14,179
	102-104		377	2,646	3,245	11,452	1,830	12,547	2,217	10,406	11,434	12,547
	189-191		689	5,095	6,092	16,457	1,821	22,907	2,557	19,338	21,023	22,907
	192-194		650	6,146	7,156	17,699	1,810	24,824	2,768	21,102	23,375	24,824
		40.00										
500-unit hypothetical problems	108-110		783	3,601	7,018	22,643	3,068	18,644	4,085	23,888	25,889	18,644
	111-113		772	11,716	7,481	26,156	3,050	33,657	3,830	24,924	26,658	33,657
	120-122		803	7,391	5,678	17,220	3,057	27,181	4,642	20,436	23,859	27,181
	135-137		827	7,999	16,418	43,061	3,052	27,715	4,578	48,923	53,221	27,715
	141-143		859	13,300	10,802	30,129	3,051	35,538	4,763	34,699	39,625	35,538
	144-146		853	13,448	11,723	32,933	3,056	35,098	4,917	38,667	42,104	35,098
	150-152		844	18,456	20,623	49,090	3,046	39,553	5,102	63,527	68,900	39,553
	153-155		842	4,923	21,080	48,495	3,043	21,923	5,078	62,263	65,672	21,923
	159-161		433	4,020	26,466	63,985	3,044	15,425	5,018	76,663	80,871	15,425
	168-170		466	4,708	7,377	22,199	3,055	15,822	5,005	25,559	28,954	15,822

Table 4: Solution times and optimality gaps for the 5 real and 60 hypothetical problems (best times and gaps are underlined)

	Test problems	No. of stands	Amax (ha)	Planning periods	Benchmark models (seconds/%) PATH		CLUSTER		BUCKET		Proposed model (s./%) ST PATH	
Real problems	NBCL5,	5,224	21.00	4X10yrs	11.23	0.03%	21.27	0.04%	532.14	0.04%	9.86	0.03%
	Canada		30.00		22.63	0.03%	86.63	0.03%	Timeout	0.07%	25.19	0.05%
			40.00		79.78	0.03%	12,747.57	0.04%	19,515.86	0.05%	51.75	0.04%
	El Dorado,	1,363	48.56	5X5yrs	20.16	0.04%	32.23	0.04%	Timeout	0.08%	60.81	0.03%
	California		60.70		75.61	0.04%	115.92	0.04%	Timeout	0.14%	27.38	0.03%
			72.84		3,518.24	0.04%	530.95	0.04%	Timeout	0.56%	1,898.16	0.05%
	Shulkell,	1,019	40.00	5X5yrs	4.30	0.03%	53.44	0.03%	133.28	0.05%	9.66	0.04%
	Nova Scotia		60.00		52.56	0.04%	7315.89	0.04%	3,339.63	0.03%	54.75	0.04%
	Kittaning4	32			8.92	0.03%	473.14	0.02%	2,724.51	0.05%	11.22	0.02%
	FivePoints	90	50.00	5X10yrs	0.56	0.05%	461.71	0.05%	7,074.25	0.03%	1.77	0.03%
	PhyllisLeeper	89			Timeout	0.0816%	Timeout	0.1602%	Timeout	0.1145%	Timeout	0.0507%
	BearTown	71			Timeout	0.1219%	Timeout	0.2408%	Timeout	0.1389%	Timeout	0.1153%
300-unit hypothetical problems	75				3.19	0.05%	54.99	0.05%	74.30	0.04%	25.09	0.04%
	76				36.97	0.05%	57.49	0.04%	562.35	0.05%	37.38	0.04%
	77				89.02	0.04%	776.26	0.05%	25.94	0.05%	24.86	0.05%
	81				132.14	0.04%	114.15	0.05%	48.50	0.05%	37.08	0.04%
	82				116.84	0.04%	239.24	0.05%	156.14	0.05%	34.59	0.05%
	83				28.98	0.05%	69.74	0.04%	144.34	0.05%	32.11	0.05%
	87				60.63	0.04%	92.27	0.03%	1,996.19	0.05%	149.17	0.04%
	88				37.80	0.04%	69.89	0.05%	228.75	0.05%	41.34	0.05%
	89				39.19	0.05%	66.50	0.03%	324.80	0.05%	20.22	0.04%
	90				59.48	0.04%	223.67	0.05%	67.78	0.05%	23.05	0.05%
	91	300	40.00	6X10yrs	80.55	0.04%	26.81	0.05%	211.03	0.05%	92.26	0.05%
	92				30.55	0.05%	44.86	0.05%	156.03	0.05%	20.86	0.05%
	93				35.44	0.05%	45.46	0.05%	18,050.93	0.05%	59.70	0.05%
	94				32.17	0.04%	30.77	0.05%	1,450.31	0.05%	5.70	0.02%
	95				113.31	0.05%	209.05	0.05%	326.24	0.05%	177.39	0.05%
	96				353.91	0.04%	96.63	0.05%	126.26	0.04%	196.33	0.04%
	97				64.64	0.05%	120.78	0.04%	141.94	0.05%	114.61	0.05%
	98				136.31	0.04%	105.19	0.04%	395.10	0.04%	50.97	0.04%
	99				40.84	0.03%	112.40	0.05%	356.22	0.05%	114.02	0.04%
	100				23.75	0.05%	136.37	0.03%	268.34	0.05%	107.64	0.05%
	101				71.47	0.05%	121.42	0.05%	266.16	0.04%	29.13	0.05%
	102				84.20	0.05%	53.36	0.04%	95.63	0.05%	17.58	0.05%

Table 4, continuation:

Test problems		No. of stands	Amax (ha)	Planning periods	Benchmark models (seconds/%) PATH		CLUSTER		BUCKET		Proposed model (s./%) ST PATH	
300-unit hypothetical problems	103				19.50	0.04%	138.49	0.05%	398.49	0.05%	29.86	0.05%
	104				18.02	0.04%	54.74	0.05%	33.56	0.05%	17.56	0.05%
	189				992.13	0.05%	566.85	0.05%	208.19	0.02%	207.11	0.03%
	190	300	40.00	6X10yrs	72.25	0.05%	192.18	0.05%	1,282.96	0.04%	159.22	0.05%
	191				61.19	0.05%	349.40	0.05%	686.80	0.05%	200.41	0.05%
	192				156.11	0.05%	171.94	0.05%	104.42	0.05%	155.69	0.05%
	193				81.34	0.05%	92.80	0.05%	495.63	0.05%	36.63	0.05%
	194				101.02	0.04%	148.93	0.05%	217.95	0.05%	59.02	0.04%
500-unit hypothetical problems	108				19.95	0.05%	130.93	0.04%	224.56	0.04%	21.22	0.05%
	109				79.76	0.03%	101.96	0.04%	93.17	0.03%	100.06	0.05%
	110				107.83	0.05%	198.51	0.05%	486.71	0.05%	65.67	0.04%
	111				49.20	0.02%	203.43	0.01%	177.02	0.05%	76.72	0.05%
	112				28.20	0.05%	125.52	0.05%	409.99	0.04%	57.33	0.04%
	113				92.17	0.03%	173.60	0.05%	295.74	0.04%	89.28	0.04%
	120				25.27	0.05%	47.80	0.05%	138.97	0.05%	26.34	0.05%
	121				76.50	0.04%	174.98	0.05%	238.63	0.04%	44.92	0.05%
	122				46.50	0.05%	60.02	0.05%	211.38	0.05%	113.75	0.03%
	135				224.38	0.05%	258.32	0.05%	196.00	0.05%	26.28	0.05%
	136				315.06	0.03%	290.56	0.04%	114.20	0.05%	109.25	0.05%
	137				166.44	0.05%	278.64	0.03%	458.33	0.05%	197.39	0.04%
	141				12.11	0.04%	228.23	0.03%	228.63	0.04%	9.89	0.04%
	142				221.72	0.04%	251.43	0.03%	89.91	0.03%	102.33	0.03%
	143	500	40.00	6X10yrs	163.59	0.05%	236.79	0.04%	152.03	0.05%	237.19	0.05%
	144				99.13	0.03%	157.24	0.05%	67.55	0.02%	465.30	0.02%
	145				106.81	0.05%	121.08	0.05%	238.72	0.05%	105.19	0.03%
	146				13.47	0.05%	124.75	0.04%	119.31	0.04%	14.84	0.03%
	150				328.42	0.05%	329.98	0.05%	116.67	0.05%	412.75	0.03%
	151				21.59	0.04%	273.07	0.04%	80.81	0.05%	22.08	0.04%
	152				187.36	0.05%	443.17	0.04%	241.78	0.04%	217.25	0.04%
	153				305.83	0.05%	530.72	0.05%	116.55	0.05%	346.91	0.03%
	154				318.42	0.05%	419.11	0.05%	141.34	0.04%	560.34	0.04%
	155				221.06	0.04%	306.70	0.03%	75.19	0.05%	121.63	0.05%
	159				397.55	0.04%	388.56	0.05%	172.63	0.04%	24.95	0.04%
	160				414.25	0.05%	477.35	0.03%	337.38	0.05%	39.33	0.05%
	161				253.06	0.05%	805.30	0.04%	240.11	0.05%	337.95	0.05%
	168				77.16	0.04%	82.71	0.05%	257.50	0.05%	46.73	0.05%
	169				119.66	0.05%	80.99	0.04%	225.45	0.04%	123.19	0.04%
	170				96.47	0.04%	128.96	0.04%	816.71	0.04%	8.13	0.05%

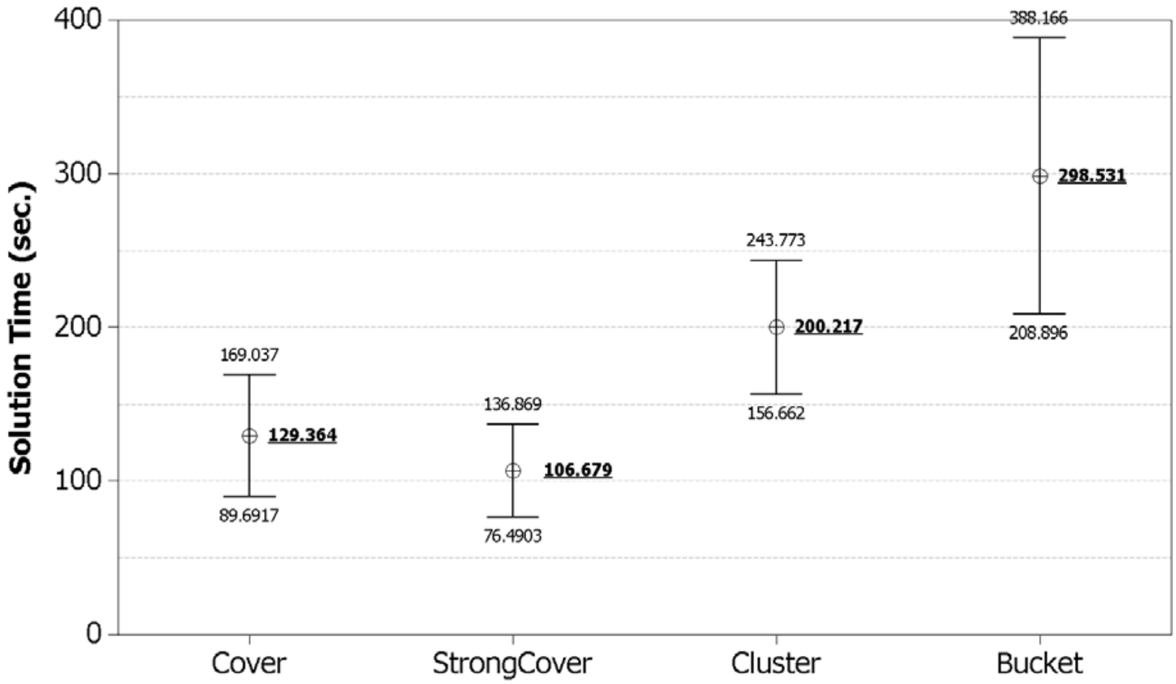

Figure 4: Confidence Intervals for Mean Solution Times

the Cluster and the Bucket models (Fig. 4). The average solution time savings afforded by the Strengthened Model relative to the Path Model were statistically insignificant.

In Table 5, we report the objective function values for the linear programming relaxations and the best integer solutions for each problem instance. The best LP bounds are shown underlined. The *root gaps*, the percentage gaps between the best integer solutions that were found by the four models and the LP relaxations, are also listed to give the reader and idea of the strength of the formulations. The smaller the root gaps, i.e., the closer the objective values of the LP relaxations to the true optima, the fewer branches might be necessary in the branch-and-bound algorithm to find a solution with a desired optimality gap (c.f., Wolsey 1998). In 30 of the 72 problem instances, the four models all provided the same LP bounds (Table 5). The Cluster Model led to strictly the lowest LP relaxations in 9 cases, while the Strengthened Path Models achieved the same in 2 cases suggesting that the Cluster Model cannot be considered "tighter" than the new model. On the other hand, the Path and the Bucket models never resulted in strictly better LP bounds than the other 2 models. This illustrates Goycoolea et al.'s (2009) and Martin et al.'s (2011) theoretical findings that the Cluster Model provides a tighter approximation of the integral convex hull of the ARM than the Path or the Bucket models. There were 57 problems where the Path, the Cluster and the Strengthened Path formulations all produced the same LP bounds. In the remaining 15 cases, the Cluster and the Strengthened Path were the best. Finally, the fact that in 14 cases (19.4% of total) the Strengthened Model led to better LP bounds than the Path Model confirms the expected empirical benefits of **Proposition 1.**

It is important to discuss the implications of the fact that the strengthening procedure itself has an associated computational cost. In our experiment, the runtime of the procedure ranged from seconds for the smaller problems (Kittaning4, FivePoints, PhyllisLeeper and Beartown) to more than an hour for the 5,224 stand NBCL5. It is clear that the computational effort must be in line with the expected benefits in solution time for this method to be worthwhile. In the case of the exceptionally hard PhyllisLeeper for example, investing an extra second in generating the stronger constraints appears to be a reasonable choice given the additional $1,924 that come from a better quality solution (Table 5). In the case of NBCL5 on the other hand, spending an extra hour on the strengthening procedure is not in line with the seconds it takes to solve this problem to the same 0.05% optimality gap with either the Path or the Strengthened Path approach. In sum, we recommend the proposed strengthening procedure for small

Table 5: Objective values for LP relaxations and best integer solutions, and root LP gaps (best LP bounds are underlined)

	Test problems	Benchmark models (LP relaxation obj. values / best integer obj. values / root LP gap %)									Proposed model		
		PATH			CLUSTER			BUCKET			ST PATH (LP relax. / best int. /%)		
Real problems	NBCL5,	102,323,350	102,254,467	0.056%	102,276,050	102,249,198	0.010%	102,417,458	102,266,014	0.148%	102,291,940	102,254,835	0.025%
	Canada	102,742,156	102,698,448	0.026%	102,721,808	102,698,360	0.007%	102,864,370	102,715,010	0.145%	102,738,668	102,680,729	0.023%
		102,919,522	102,882,644	0.018%	102,909,021	102,867,730	0.008%	102,986,688	102,900,889	0.083%	102,918,466	102,876,810	0.017%
	El Dorado,	2,727,376	2,725,625	0.061%	2,726,748	2,725,591	0.038%	2,730,503	2,705,498	0.175%	2,726,877	2,725,716	0.043%
	California	2,731,735	2,729,819	0.051%	2,731,385	2,729,935	0.038%	2,735,609	2,692,802	0.193%	2,731,506	2,730,338	0.043%
		2,736,140	2,733,759	0.087%	2,735,843	2,733,659	0.076%	2,738,128	2,711,791	0.160%	2,736,078	2,733,496	0.085%
	Shulkell,	85,063,615	85,039,452	0.028%	85,060,663	85,031,921	0.025%	85,062,462	85,006,172	0.027%	85,060,914	85,024,839	0.025%
	Nova Scotia	85,095,790	85,061,135	0.025%	85,095,658	85,065,819	0.025%	85,105,213	85,074,316	0.036%	85,095,790	85,061,135	0.025%
	Kittaning4	2,276,234	2,255,947	0.891%	2,276,234	2,255,947	0.891%	2,324,163	2,255,947	2.935%	2,276,234	2,255,947	0.891%
	FivePoints	2,608,121	2,606,852	0.029%	2,608,070	2,606,954	0.027%	2,704,523	2,607,195	3.592%	2,608,070	2,607,371	0.027%
	PhyllisL'pr	7,482,722	7,476,488	0.058%	7,482,394	7,470,196	0.053%	7,482,769	7,473,777	0.058%	7,482,716	7,478,412	0.058%
	BearTown	6,327,543	6,307,133	0.314%	6,327,543	6,300,412	0.314%	6,327,916	6,307,018	0.320%	6,327,543	6,307,673	0.314%
300-unit hypothetical problems	75	14,182,313	14,175,561	0.038%	14,182,313	14,174,792	0.038%	14,182,313	14,175,408	0.038%	14,182,313	14,176,942	0.038%
	76	14,171,086	14,162,890	0.036%	14,171,086	14,164,760	0.036%	14,174,886	14,165,934	0.063%	14,171,086	14,164,238	0.036%
	77	14,173,397	14,166,075	0.037%	14,173,397	14,166,292	0.037%	14,175,757	14,168,158	0.054%	14,173,397	14,165,398	0.037%
	81	14,193,527	14,186,234	0.051%	14,193,527	14,186,215	0.051%	14,193,527	14,185,905	0.051%	14,193,527	14,186,242	0.051%
	82	14,191,372	14,184,206	0.051%	14,191,372	14,183,724	0.051%	14,191,372	14,183,763	0.051%	14,191,372	14,184,054	0.051%
	83	14,128,707	14,120,741	0.049%	14,128,707	14,121,831	0.049%	14,128,707	14,120,591	0.049%	14,128,707	14,121,578	0.049%
	87	14,391,782	14,385,344	0.032%	14,391,782	14,387,165	0.032%	14,391,782	14,384,426	0.032%	14,391,782	14,386,173	0.032%
	88	14,100,029	14,094,012	0.043%	14,100,029	14,092,137	0.043%	14,100,029	14,091,465	0.043%	14,100,029	14,093,212	0.043%
	89	14,099,391	14,092,598	0.034%	14,099,391	14,094,595	0.034%	14,099,391	14,092,347	0.034%	14,099,391	14,093,598	0.034%
	90	14,181,504	14,173,661	0.022%	14,181,504	14,173,301	0.022%	14,186,049	14,178,334	0.054%	14,181,504	14,173,604	0.022%
	91	14,204,113	14,196,652	0.053%	14,204,113	14,196,624	0.053%	14,204,113	14,196,447	0.053%	14,204,113	14,196,466	0.053%
	92	14,179,482	14,171,813	0.054%	14,179,482	14,171,389	0.054%	14,179,482	14,171,284	0.054%	14,179,482	14,171,438	0.054%
	93	14,234,467	14,225,459	0.049%	14,232,858	14,225,973	0.037%	14,235,408	14,227,534	0.055%	14,232,858	14,225,776	0.037%
	94	14,295,642	14,287,972	0.033%	14,293,710	14,286,841	0.020%	14,296,898	14,289,600	0.042%	14,293,710	14,290,891	0.020%
	95	14,121,686	14,113,814	0.056%	14,121,686	14,113,770	0.059%	14,121,686	14,113,705	0.056%	14,121,686	14,113,706	0.056%
	96	14,216,763	14,209,489	0.038%	14,216,763	14,209,135	0.038%	14,217,397	14,211,321	0.043%	14,216,763	14,209,739	0.038%
	97	14,013,484	14,006,987	0.042%	14,013,484	14,007,639	0.042%	14,013,484	14,006,600	0.042%	14,013,484	14,006,503	0.042%
	98	14,357,591	14,351,592	0.039%	14,357,591	14,351,628	0.039%	14,357,591	14,351,856	0.039%	14,357,591	14,352,052	0.039%
	99	14,226,108	14,222,512	0.025%	14,226,108	14,219,383	0.025%	14,230,787	14,222,334	0.058%	14,226,108	14,220,314	0.025%
	100	14,177,017	14,170,430	0.009%	14,177,017	14,172,252	0.009%	14,182,818	14,175,795	0.050%	14,177,017	14,170,009	0.009%
	101	14,156,463	14,148,681	0.055%	14,156,463	14,148,245	0.055%	14,156,853	14,148,716	0.058%	14,156,463	14,148,340	0.055%
	102	14,214,355	14,207,649	0.042%	14,214,355	14,208,428	0.042%	14,214,355	14,206,794	0.042%	14,214,355	14,207,415	0.042%

Table 5, continuation:

	Test problems	Benchmark models (LP relaxation obj. values / best integer obj. values / root LP gap %)									Proposed model		
		PATH			CLUSTER			BUCKET			ST PATH (LP relax. / best int. /%)		
300-unit hypothetical problems	103	14,143,274	14,138,033	0.0231%	14,143,274	14,136,341	0.0231%	14,147,050	14,140,006	0.0498%	14,143,274	14,136,236	0.0231%
	104	14,185,694	14,178,943	0.0476%	14,185,694	14,178,028	0.0476%	14,186,308	14,178,780	0.0519%	14,185,694	14,177,992	0.0476%
	189	13,689,576	13,682,263	0.0276%	13,689,576	13,681,919	0.0276%	13,690,835	13,685,802	0.0368%	13,689,576	13,684,157	0.0276%
	190	13,714,667	13,706,776	0.0456%	13,714,667	13,707,100	0.0456%	13,715,575	13,708,416	0.0522%	13,714,667	13,706,571	0.0456%
	191	13,698,038	13,690,271	0.0567%	13,698,038	13,690,079	0.0567%	13,698,038	13,690,165	0.0567%	13,698,038	13,689,600	0.0567%
	192	13,710,587	13,703,751	0.0458%	13,710,587	13,703,740	0.0458%	13,710,587	13,704,312	0.0458%	13,710,587	13,703,743	0.0458%
	193	13,663,844	13,657,173	0.0473%	13,663,844	13,657,055	0.0473%	13,663,844	13,657,046	0.0473%	13,663,844	13,657,378	0.0473%
	194	13,724,074	13,716,740	0.0534%	13,724,074	13,716,097	0.0534%	13,724,073	13,716,368	0.0534%	13,724,074	13,716,531	0.0534%
500-unit hypothetical Problems	108	23,690,743	23,679,014	0.0409%	23,690,743	23,680,679	0.0409%	23,691,668	23,681,045	0.0448%	23,690,743	23,679,124	0.0409%
	109	23,650,556	23,642,740	0.0244%	23,650,556	23,639,525	0.0244%	23,651,307	23,644,785	0.0276%	23,650,556	23,637,753	0.0244%
	110	23,906,452	23,895,364	0.0367%	23,906,452	23,894,138	0.0367%	23,906,452	23,894,748	0.0367%	23,906,452	23,897,676	0.0367%
	111	22,530,821	22,526,789	0.0122%	22,530,821	22,528,083	0.0122%	22,531,350	22,520,040	0.0145%	22,530,821	22,519,788	0.0122%
	112	22,543,397	22,531,500	0.0094%	22,542,843	22,531,154	0.0069%	22,550,230	22,541,276	0.0397%	22,542,843	22,532,126	0.0069%
	113	22,523,724	22,516,624	0.0315%	22,523,724	22,512,572	0.0315%	22,523,724	22,514,151	0.0315%	22,523,724	22,515,752	0.0315%
	120	26,632,845	26,618,613	0.0391%	26,632,845	26,618,077	0.0391%	26,635,110	26,622,421	0.0476%	26,631,260	26,618,239	0.0332%
	121	26,577,471	26,565,992	0.0386%	26,577,471	26,564,388	0.0386%	26,577,471	26,567,221	0.0386%	26,577,471	26,564,390	0.0386%
	122	26,627,877	26,613,338	0.0387%	26,627,877	26,613,012	0.0387%	26,628,954	26,614,405	0.0428%	26,626,227	26,617,566	0.0325%
	135	20,889,919	20,879,572	0.0479%	20,889,919	20,879,917	0.0479%	20,889,919	20,878,540	0.0479%	20,889,919	20,879,860	0.0479%
	136	20,872,885	20,865,486	0.0355%	20,872,885	20,863,270	0.0355%	20,872,885	20,862,552	0.0355%	20,872,885	20,862,539	0.0355%
	137	20,880,030	20,870,338	0.0322%	20,880,030	20,873,297	0.0322%	20,880,686	20,869,499	0.0354%	20,880,030	20,872,695	0.0322%
	141	22,846,718	22,836,563	0.0280%	22,846,718	22,840,327	0.0280%	22,846,964	22,836,376	0.0291%	22,846,718	22,837,461	0.0280%
	142	22,840,611	22,828,511	0.0231%	22,840,611	22,832,017	0.0231%	22,841,245	22,835,332	0.0259%	22,839,858	22,831,283	0.0198%
	143	22,824,833	22,813,725	0.0439%	22,824,833	22,814,820	0.0439%	22,824,833	22,814,500	0.0439%	22,824,833	22,814,550	0.0439%
	144	22,875,066	22,868,289	0.0203%	22,875,066	22,864,260	0.0203%	22,875,066	22,870,426	0.0203%	22,875,066	22,868,853	0.0203%
	145	22,845,542	22,834,281	0.0330%	22,845,542	22,834,180	0.0330%	22,848,531	22,836,454	0.0461%	22,845,542	22,837,995	0.0330%
	146	22,883,358	22,872,574	0.0321%	22,883,358	22,873,527	0.0321%	22,883,358	22,873,330	0.0321%	22,883,358	22,876,020	0.0321%
	150	20,886,989	20,877,160	0.0294%	20,886,989	20,876,644	0.0294%	20,886,988	20,877,210	0.0294%	20,886,989	20,880,838	0.0294%
	151	20,919,065	20,910,856	0.0392%	20,919,065	20,910,674	0.0392%	20,919,065	20,908,807	0.0392%	20,919,065	20,910,734	0.0392%
	152	20,892,527	20,882,298	0.0428%	20,892,527	20,883,231	0.0428%	20,892,527	20,883,575	0.0428%	20,892,527	20,883,308	0.0428%
	153	20,923,910	20,913,628	0.0325%	20,923,910	20,913,632	0.0325%	20,923,910	20,913,673	0.0325%	20,923,910	20,917,115	0.0325%
	154	20,912,932	20,902,593	0.0395%	20,912,932	20,902,817	0.0395%	20,912,932	20,903,733	0.0395%	20,912,932	20,904,667	0.0395%
	155	20,898,163	20,889,153	0.0314%	20,898,163	20,891,592	0.0314%	20,898,745	20,888,649	0.0342%	20,898,163	20,888,331	0.0314%
	159	19,702,157	19,693,333	0.0266%	19,702,157	19,692,761	0.0266%	19,704,538	19,696,916	0.0387%	19,702,157	19,695,050	0.0266%
	160	19,698,736	19,688,953	0.0278%	19,698,736	19,693,258	0.0278%	19,700,927	19,690,925	0.0389%	19,698,736	19,689,095	0.0278%
	161	19,621,571	19,611,986	0.0369%	19,621,571	19,614,297	0.0369%	19,623,727	19,614,327	0.0479%	19,621,571	19,611,832	0.0369%
	168	24,813,347	24,801,626	0.0451%	24,813,347	24,802,158	0.0451%	24,814,142	24,802,146	0.0483%	24,813,347	24,801,082	0.0451%
	169	24,845,788	24,831,994	0.0245%	24,845,788	24,834,503	0.0245%	24,849,053	24,839,712	0.0376%	24,845,788	24,834,591	0.0245%
	170	24,802,395	24,791,551	0.0374%	24,802,395	24,792,985	0.0374%	24,802,395	24,793,127	0.0374%	24,802,395	24,790,591	0.0374%
Means		22,293,538		0.0536%	22,292,330		0.0514%	22,300,549		0.1449%	22,292,878		0.0520%

problems that are very hard to solve with other models.

Another point that needs to be made with respect to the strengthening concept is that the spatial structure of a forest planning problem can have an impact on the number of extension and lifting opportunities in a path/cover-based formulation. The more cover constraints exist that can be strengthened in a problem, the more likely it is that the procedure can lead to reduced solution times. However, while many cover constraints may be extended and lifted in some problems, none will be possible in others. The average size of the management units relative to the maximum harvest opening size, as well as the average number of adjacent units per unit (vertex degree) have an effect on the strengthening potential of the proposed procedure. The smaller the average size of the units relative to the cut limit, the less likely it is that the cover inequalities can be strengthened. This is because forests that have smaller units relative to the maximum harvest opening size will give rise to covers that comprise more units. Covers that comprise more units are harder to strengthen because the smaller units that are adjacent to the covers are less likely to satisfy **Proposition 1**. This is the reason why the amount by which the strengthened formulation reduced the LP bounds relative to those produced by the Path formulation for NBCL5 diminishes as the harvest size limit increases from 21 to 40 ha. The root gap is 0.0545% with the strengthened approach at the 21 ha opening size vs. the 0.0852% with the original path, but it is only 0.0636% at the 30ha level vs. the 0.0670%, and it is 0.0725% vs. 0.0735% at the 40 ha level (Table 5). In other words, it is no surprise that the extent to which the root gaps are reduced by the strengthened formulation becomes smaller and smaller as the harvest opening size increases. Similarly, a higher vertex degree is likely to increase the number of candidate stands that are adjacent to more than one member of the cover. Thus, lifting opportunities are more likely to occur when there are more adjacencies. Clearly, a close inspection of the spatial configuration of the management units in the forest in question can be very helpful to decide if the strengthening procedure should be employed or not.

6 Conclusions

In this article, we showed how the path/cover constraints, generated by McDill et al.'s (2002) Path Algorithm for area-based forest planning models, can be strengthened. We provided both theoretical and empirical evidence (see Table 5) that the proposed constraints are indeed stronger than the original ones. We also showed that the strengthened formulation can outperform the other models computationally in many cases. As a caveat, however, we emphasize that there is a computational cost associated with the strengthening procedure, which must be offset by solution timesavings if the new model is to be used efficiently. A preliminary analysis of the spatial configuration of the management units in the landscape could help the analyst determine whether it would likely be worthwhile to apply the strengthening procedure. Lastly, we mention that there are many additional strategies that could be followed to improve the use of the proposed concept in practice. First, not all strengthened cover inequalities might need to be generated, but only those that cut off fractional solutions. This observation could lead to a cutting plane algorithm where the strengthened cuts are only created and applied if they have the potential to cut off fractional solutions. Second, it was shown that in certain cases stronger inequalities might exist than those generated by the proposed algorithm. Finding ways to generate these stronger cuts efficiently could further reduce solution times. It is also important to point out that we did not compare the proposed approaches with Crowe et al.'s (2003) ARM cliques or Gunn and Richards' (2005) stand-centered constraints. It is possible that some combination of cover constraints, strengthened cover constraints, ARM cliques, or stand-centered constraints would provide superior computational results.

Finally, we also note that the computational experiments presented in this paper are the most extensive to date in the area-based adjacency literature. We solved 60 hypothetical and 12 real instances including both small and large problems, with planning horizons of varying lengths, with varying vertex degrees and with different forest types and age classes. While the computational results are not conclusive with respect to the real problems, our data should serve as good reference point for readers who like to know what to expect from these alternative models.

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